Interference of Waves in Two Dimensions 9.3

Constructive and **destructive interference** may occur in two dimensions, sometimes producing fixed patterns of interference. To produce a fixed pattern, the interfering waves must have the same frequency (and thus the same wavelength) and also similar amplitudes. Standing waves in a string or rope, fixed at one end, illustrate interference in one dimension. Patterns of interference also occur between two identical waves when they interfere in a two-dimensional medium such as the water in a ripple tank.

Figure 1 shows two point sources vibrating with identical frequencies and amplitudes and in phase. As successive crests and troughs travel out from the two sources, they interfere with each other, sometimes crest on crest, sometimes trough on trough, and sometimes crest on trough, producing areas of constructive and destructive interference.

constructive interference occurs when waves build each other up, producing a resultant wave of greater amplitude than the given waves

destructive interference occurs when waves diminish one another, producing a resultant wave of lower amplitude than the given waves

nodal line a line of destructive interference

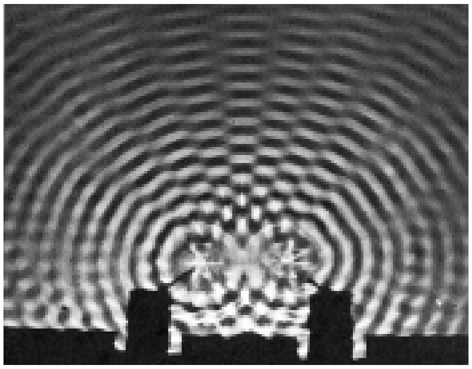
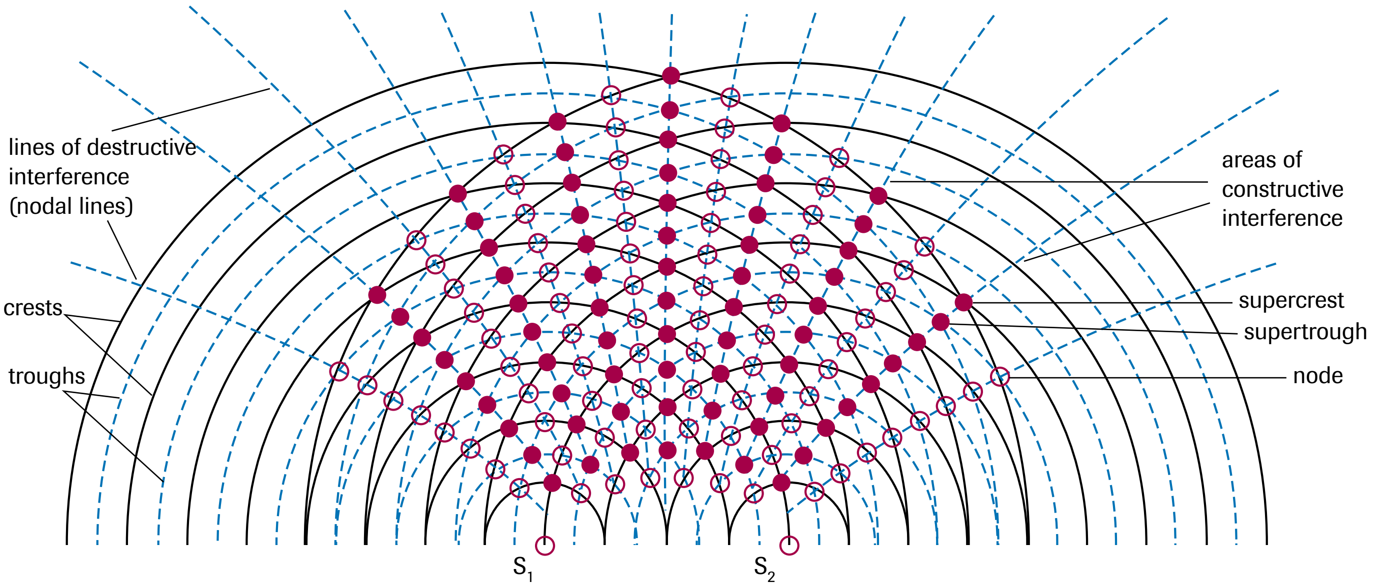


Figure 1
Interference between two point sources in phase in a ripple tank

You can see from **Figure 1** that these areas spread out from the source in symmetrical patterns, producing **nodal lines** and areas of constructive interference. When illuminated from above, the nodal lines appear on the surface below the water in the ripple tank as stationary grey areas. Between the nodal lines are areas of constructive interference that appear as alternating bright (double-crest) and dark (double-trough) lines of constructive interference. You can see these alternating areas of constructive and destructive interference in **Figure 2**. Although the nodal lines appear to be straight, their paths from the sources are actually hyperbolas.

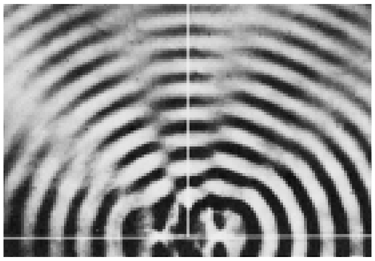
Figure 2
The interference pattern between two identical sources (S_1 and S_2), vibrating in phase, is a symmetric pattern of hyperbolic lines of destructive interference (nodal lines) and areas of constructive interference.



> **LEARNING TIP**
>
> **In Phase versus Out of Phase**
> Recall from previous studies that objects vibrating in phase have the same periods and pass through the rest point at the same time. Objects vibrating out of phase may not have the same period, but if they do, they do not pass through the rest point at the same time.

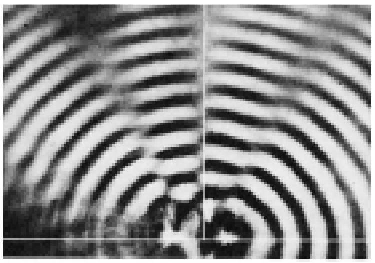
This symmetrical pattern remains stationary, provided three factors do not change: the frequency of the two sources, the distance between the sources, and the relative phase of the sources. When the frequency of the sources is increased, the wavelength decreases, bringing the nodal lines closer together and increasing their number. If the distance between the two sources is increased, the number of nodal lines also increases. As you would expect, neither of these factors changes the symmetry of the pattern: provided the two sources continue to be in phase, an area of constructive interference runs along the right bisector, and equal numbers of nodal lines appear on the two sides of the right bisector. If other factors are kept constant, but the relative phase of the two sources changes, the pattern shifts (as in **Figure 3**), with the number of nodal lines remaining the same. For example, if S_1 is delayed, the pattern shifts to the left of the right bisector.

(a)



S_1 S_2

(b)



S_1 S_2

Figure 3
The effect of a phase delay on the interference pattern for two point sources. In **(a)** the sources are in phase; in **(b)** the phase delay is 180°.

Mathematical Analysis of the Two-Point-Source Interference Pattern

The two-point-source interference pattern is useful because it allows direct measurement of the wavelength (since it is easy to keep the interference pattern relatively stationary). By taking a closer look at the two-point-source interference pattern, we can develop some mathematical relationships that will be useful in Section 9.5 and the next chapter for analyzing the interference of other kinds of waves.

Consider the ripple-tank interference pattern produced by a pair of identical point sources S_1 and S_2, vibrating in phase and separated by three wavelengths. In this pattern, there are an equal number of nodal lines on either side of the right bisector. These lines are numbered 1 and 2 on both sides of the right bisector (**Figure 4**). (So, for example, there are two nodal lines labelled $n = 1$: the first line on either side of the right bisector.) If we take a point P_1 on one of the first nodal lines and connect it to each of the two sources by the lines P_1S_1 and P_1S_2 as in **Figure 4**, we might find that $P_1S_1 = 4\lambda$ and $P_1S_2 = \frac{7}{2}\lambda$. The difference between these two distances, called the **difference in path length**, is

$$|P_1S_1 - P_1S_2| = \frac{1}{2}\lambda$$

This relationship holds for any point on the first nodal line on either side of the right bisector. (We take absolute values when expressing the difference in path length because our only interest is in the size of the discrepancy in lengths. We do not care which length is the greater of the two.) When we measure in the same way the difference in path length for any point P_2 on a nodal line second from the centre, we find that

$$|P_2S_1 - P_2S_2| = \frac{3}{2}\lambda$$

Continuing this procedure, we can arrive at a general relationship for any point P on the nth nodal line:

$$|P_nS_1 - P_nS_2| = \left(n - \frac{1}{2}\right)\lambda \qquad \text{Equation (1)}$$

difference in path length in an interference pattern, the absolute value of the difference between the distance of any point P from one source and the distance of the same point P from the other source: $|P_1S_1 - P_1S_2|$

You can use this relationship to find the wavelength of interfering waves in the ripple tank by locating a point on a specific nodal line, measuring the path lengths, and substituting in Equation (1).

If the wavelengths are too small or the point P is too far away from the two sources, the difference in path length is too small to be measured accurately. To handle these two cases (which could occur either individually or together), we need another technique.

For any point P_n, the difference in path length is the distance AS_1 in **Figure 5(a)**:

$$|P_nS_1 - P_nS_2| = AS_1$$

Figure 5(b) shows that when P_n is very far away compared to the separation d of the two sources, the lines P_nS_1 and P_nS_2 are very nearly parallel; that is, as $P_n \rightarrow \infty$, P_nS_1 and P_nS_2 become nearly parallel. In this case, the line AS_2 forms a right angle with both of these lines, as illustrated, making the triangle S_1S_2A a right-angled triangle (**Figure 5(b)**). Therefore, the difference in path length can be expressed in terms of the sine of the angle θ_n:

$$\sin\theta_n = \frac{AS_1}{d}$$

$$AS_1 = d\sin\theta_n \qquad \text{Equation (2)}$$

But $AS_1 = P_nS_1 - P_nS_2$. Therefore, by combining Equations (1) and (2) we get

$$d\sin\theta_n = \left(n - \frac{1}{2}\right)\lambda$$

$$\sin\theta_n = \left(n - \frac{1}{2}\right)\frac{\lambda}{d}$$

where θ_n is the angle for the nth nodal line, λ is the wavelength, and d is the distance between the sources.

This equation allows us to make a quick approximation of the wavelength for a specific interference pattern. Since $\sin\theta_n$ cannot be greater than 1, $\left(n - \frac{1}{2}\right)\frac{\lambda}{d}$ cannot be greater than 1. The largest value of n that satisfies this condition is the number of nodal lines on either side of the right bisector. Measuring d and counting the number of nodal lines gives an approximation for the wavelength. For example, if d is 2.0 m and the number of nodal lines is 4, the wavelength can be approximated as follows:

$$\sin\theta_n = \left(n - \frac{1}{2}\right)\frac{\lambda}{d}$$

Or, since the maximum possible value of $\sin\theta_n$ is 1,

$$\left(n - \frac{1}{2}\right)\frac{\lambda}{d} \approx 1$$

$$\left(4 - \frac{1}{2}\right)\frac{\lambda}{2.0\text{ m}} \approx 1$$

$$\lambda \approx 0.57\text{ cm}$$

In the ripple tank, it is relatively easy to measure the angle θ_n. The measurement is not, however, easy for light waves (Section 9.5), where both the wavelength and the distance between the sources are very small and the nodal lines are close together. We therefore seek a technique for measuring $\sin\theta_n$ without measuring θ_n itself.

We noted earlier that a nodal line is a hyperbola. But at positions on nodal lines relatively far away from the two sources, the nodal lines are nearly straight, appearing to originate from the midpoint of a line joining the two sources.

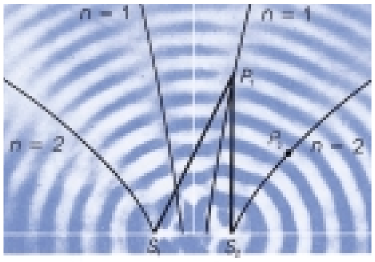


Figure 4
For any point P_1 on the first nodal line, the difference in path length from P_1 to S_1 and from P_1 to S_2 is $\frac{1}{2}\lambda$. For any point P_2 on the second nodal line, the difference in path length is $\frac{3}{2}\lambda$.

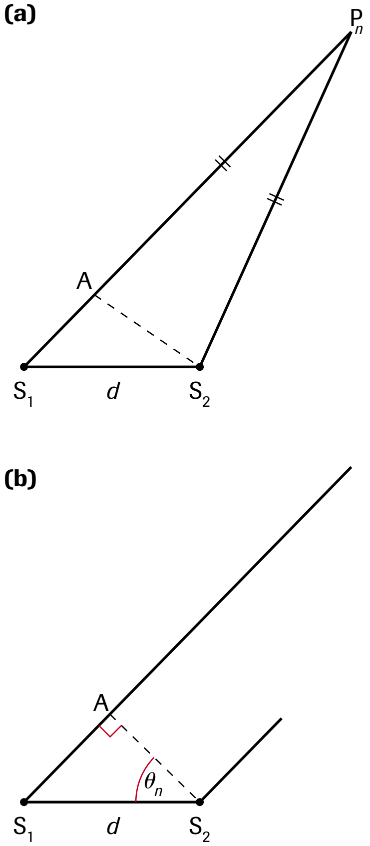


Figure 5
(a) Point P_n is near.
(b) Point P_n is far enough away that P_nS_1 and P_nS_2 are considered parallel.

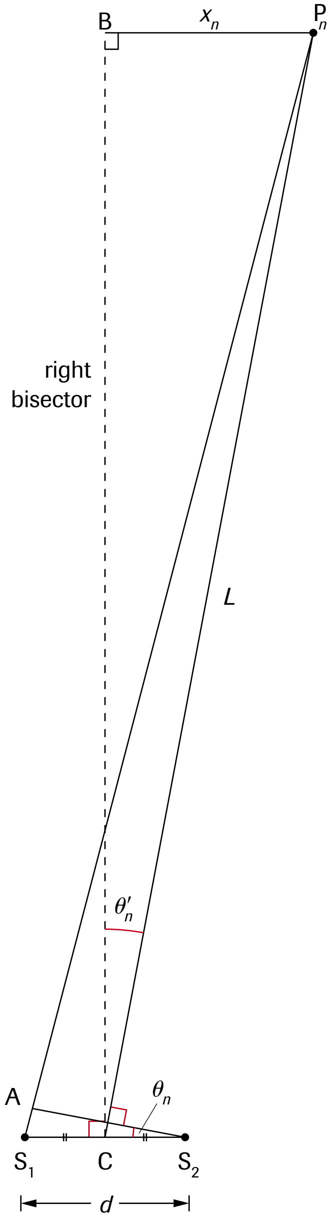


Figure 6

For a point P_n located on a nodal line, far away from the two sources, the line from P_n to the midpoint between the two sources, P_nC, is essentially parallel to P_nS_1 (**Figure 6**). This line is also perpendicular to AS_2. Since the right bisector (CB) is perpendicular to S_1S_2, we can easily show that $\theta'_n = \theta_n$ (**Figure 7**).

In **Figure 6**, sin θ'_n can be determined from the triangle P_nBC as follows:

$$\sin \theta'_n = \frac{x_n}{L}$$

$$\text{Since } \sin \theta_n = \left(n - \frac{1}{2}\right)\frac{\lambda}{d}$$

$$\text{and } \sin \theta'_n = \sin \theta_n,$$

$$\frac{x_n}{L} = \left(n - \frac{1}{2}\right)\frac{\lambda}{d}$$

In this derivation, d is the distance between the sources, x_n is the perpendicular distance from the right bisector to the point on the nodal line, L is the distance from the point P_n to the midpoint between the two sources, and n is the number of the nodal line.

Note that our derivation assumes a pair of point sources, vibrating in phase.

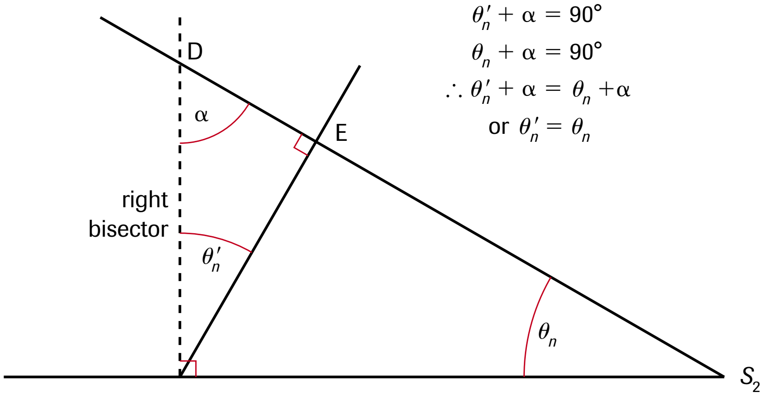


Figure 7

SAMPLE problem 1

The distance from the right bisector to the second nodal line in a two-point interference pattern is 8.0 cm. The distance from the midpoint between the two sources to point P is 28 cm. What is angle θ_2 for the second nodal line?

Solution

$x_2 = 8.0$ cm

$L = 28$ cm

$\theta_2 = ?$

$$\sin \theta_2 = \frac{x_2}{L} = \frac{8.0 \text{ cm}}{28 \text{ cm}}$$

$$\theta_2 = 16.6°, \text{ or } 17°$$

The angle θ_2 for the second nodal line is 17°.

SAMPLE problem 2

Two identical point sources 5.0 cm apart, operating in phase at a frequency of 8.0 Hz, generate an interference pattern in a ripple tank. A certain point on the first nodal line is located 10.0 cm from one source and 11.0 cm from the other. What is (a) the wavelength of the waves and (b) the speed of the waves?

Solution

$d = 5.0 \text{ cm}$	$PS_2 = 10.0 \text{ cm}$	$\lambda = ?$
$f = 8.0 \text{ Hz}$	$PS_1 = 11.0 \text{ cm}$	$v = ?$

(a)

$$\begin{aligned} |PS_1 - PS_2| &= \left(n - \frac{1}{2}\right)\lambda \\ |11.0 \text{ cm} - 10.0 \text{ cm}| &= \left(1 - \frac{1}{2}\right)\lambda \\ \lambda &= 2.0 \text{ cm} \end{aligned}$$

The wavelength of the waves is 2.0 cm.

(b)

$$\begin{aligned} v &= f\lambda \\ &= (8.0 \text{ Hz})(2.0 \text{ cm}) \\ v &= 16 \text{ cm/s} \end{aligned}$$

The speed of the waves is 16 cm/s.

Practice

Understanding Concepts

1. Two point sources, S_1 and S_2, oscillating in phase send waves into the air at the same wavelength, 1.98 m. Given that there is a nodal point where the two waves overlap, find the smallest corresponding path length difference.
2. In a ripple tank, a point on the third nodal line from the centre is 35.0 cm from one source and 42.0 cm from another. The sources are 11.2 cm apart and vibrate in phase at 10.5 Hz. Calculate the wavelength and the speed of the waves.
3. An interference pattern is set up by two point sources of the same frequency, vibrating in phase. A point on the second nodal line is 25.0 cm from one source, 29.5 cm from the other. The speed of the waves is 7.5 cm/s. Calculate the wavelength and the frequency of the sources.

Answers

1. 0.99 m
2. 2.80 cm; 29.4 cm/s
3. 3.0 cm; 2.5 Hz

Up to this point, we have discussed two-point-source wave interference in the abstract, with formulas and geometrical diagrams, and have inspected some photographs. However, we have not studied this phenomenon first hand. Investigation 9.3.1 in the Lab Activities section at the end of this chapter provides you with the opportunity to confirm the analysis of two-point-source interference in the lab.

INVESTIGATION 9.3.1

Interference of Waves in Two Dimensions (p. 482)
How can you test our analysis of two-point wave interference? What equipment will you need?

SUMMARY Interference of Waves in Two Dimensions

- A pair of identical point sources operating in phase produces a symmetrical pattern of constructive interference areas and nodal lines. The nodal lines are hyperbolas radiating from between the two sources.
- Increasing the frequency (lowering the wavelength) of the sources increases the number of nodal lines.

- Increasing the separation of the sources increases the number of nodal lines.
- Changing the relative phase of the sources changes the position of the nodal lines but not their number.
- The relationship $\sin\theta_n = \left(n - \frac{1}{2}\right)\frac{\lambda}{d}$, or $\frac{x_n}{L} = \left(n - \frac{1}{2}\right)\frac{\lambda}{d}$, can be used to solve for an unknown in a two-point-source interference pattern.

Section 9.3 Questions

Understanding Concepts

1. List three conditions necessary for a two-point-source interference pattern to remain stable.

2. By how much must path lengths differ if two waves from identical sources are to interfere destructively?

3. What ratio of $\frac{\lambda}{d}$ would produce no nodal line?

4. Explain why the interference pattern between two point sources is difficult to see
  (a) if the distance between the sources is large
  (b) if the relative phase of the two sources is constantly changing

5. Two point sources, 5.0 cm apart, are operating in phase, with a common frequency of 6.0 Hz, in a ripple tank. A metre stick is placed above the water, parallel to the line joining the sources. The first nodal lines (the ones adjacent to the central axis) cross the metre stick at the 35.0-cm and 55.0-cm marks. Each of the crossing points is 50.0 cm from the midpoint of the line joining the two sources. Draw a diagram of the tank, and then calculate the wavelength and speed of the waves.

6. Two sources of waves are in phase and produce identical waves. These sources are mounted at the corners of a square. At the centre of the square, waves from the sources produce constructive interference, no matter which two corners of the square are occupied by the sources. Explain why, using a diagram.

7. In a large water tank experiment, water waves are generated with straight, parallel wave fronts, 3.00 m apart. The wave fronts pass through two openings 5.00 m apart in a long board. The end of the tank is 3.00 m beyond the board. Where would you stand, relative to the perpendicular bisector of the line between the openings, if you want to receive little or no wave action?

8. A page in a student's notebook lists the following information, obtained from a ripple tank experiment with two point sources operating in phase: $n = 3$, $x_3 = 35$ cm, $L = 77$ cm, $d = 6.0$ cm, $\theta_3 = 25°$, and 5 crests $= 4.2$ cm. Calculate the wavelength of the waves using three methods.

9. Two very small, identical speakers, each radiating sound uniformly in all directions, are placed at points S_1 and S_2 as in **Figure 8**. The speakers are connected to an audio source in such a way that they radiate in phase, at the common wavelength of 2.00 m. Sound propagates in air at 338 m/s.
  (a) Calculate the frequency of the sound.
  (b) Point M, a nodal point, is 7.0 m from S_1 and more than 7.0 m from S_2. Find three possible distances M could be from S_2.
  (c) Point N, also a nodal point, is 12.0 m from S_1 and 5.0 m from S_2. On which nodal line is N located?

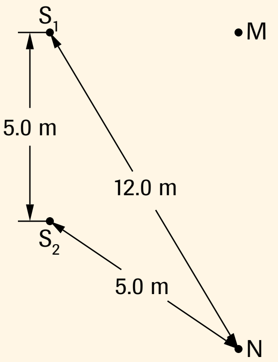


Figure 8

Applying Inquiry Skills

10. We have said that waves in a ripple tank are a "reasonable approximation" to true transverse waves.
  (a) Research the Internet or other sources and report on how the behaviour of a particle in a water wave does not exhibit strict transverse wave characteristics.

GO www.science.nelson.com

  (b) When water waves enter very shallow water, for example as they approach a beach, they not only slow down but also curl and "break." Explain this behaviour using the information you obtained in (a).
  (c) If water waves are not true transverse waves, how can we justify using them to discover the properties of transverse waves?

Making Connections

11. Two towers of a radio station are 4.00×10^2 m apart along an east–west line. The towers act essentially as point sources, radiating in phase at a frequency of 1.00×10^6 Hz. Radio waves travel at 3.00×10^8 m/s.
  (a) In which directions is the intensity of the radio signal at a maximum for listeners 20.0 km north of the transmitter (but not necessarily directly north of it)?
  (b) In which directions would you find the intensity at a minimum, north of the transmitter, if the towers were to start transmitting in opposite phase?