6.2 Orbits and Kepler's Laws

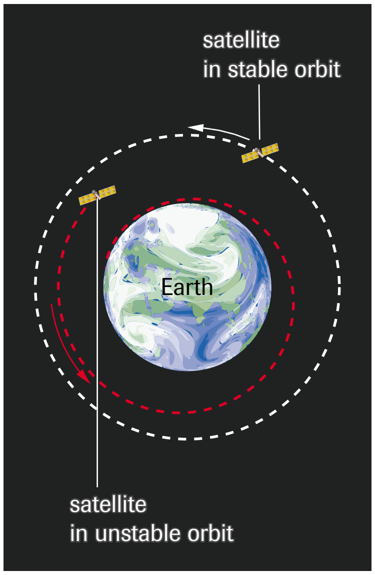

Figure 1
Comparing stable and unstable orbits of an artificial satellite. If a satellite is far enough from Earth's surface that atmospheric friction is negligible, then the firing of booster rockets is unnecessary.

Periodically we hear of "space junk" that falls through Earth's atmosphere, leaving streaks of light as friction causes it to vaporize. This "junk" falls toward Earth when its orbit becomes unstable, yet when it was in proper working order, it remained in a stable orbit around Earth (**Figure 1**). To maintain a stable orbit, a satellite or other space vehicle must maintain a required speed for a particular radius of orbit. This requires a periodic firing of the small booster rockets on the vehicle to counteract the friction of the thin atmosphere. When the Russian space station Mir was no longer needed, its booster rockets were not fired, which resulted in a gradual loss of speed because of friction. Without the correct speed to continue in a curved path that follows Earth's curvature, Mir's orbit became unstable and it was pulled down to Earth by gravity. After 15 years in service, Mir re-entered Earth's atmosphere on March 23, 2001.

We saw in Section 3.4 that for a satellite to maintain a stable circular orbit around Earth, it must maintain a specific speed v that depends on the mass of Earth and the radius of the satellite's orbit. The mathematical relationship derived for circular motion was summarized in the equation

$$v = \sqrt{\frac{Gm_E}{r}}$$

where G is the universal gravitation constant, m_E is the mass of Earth, v is the speed of the satellite, and r is the distance from the centre of Earth to the satellite.

This equation is not restricted to objects in orbit around Earth. We can rewrite the equation for any central body of mass M around which a body is in orbit. Thus, in general,

$$v = \sqrt{\frac{GM}{r}}$$

SAMPLE problem 1

Determine the speeds of the second and third planets from the Sun. Refer to Appendix C for the required data.

Solution

We will use the subscript V to represent Venus (the second planet), the subscript E to represent Earth (the third planet), and the subscript S to represent the Sun.

$G = 6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2$

$M_S = 1.99 \times 10^{30}$ kg

$r_V = 1.08 \times 10^{11}$ m

$r_E = 1.49 \times 10^{11}$ m

$v_V = ?$

$v_E = ?$

$$v_V = \sqrt{\frac{GM_S}{r_V}}$$

$$= \sqrt{\frac{(6.67 \times 10^{-11}\ \text{N}\cdot\text{m}^2/\text{kg}^2)(1.99 \times 10^{30}\ \text{kg})}{1.08 \times 10^{11}\ \text{m}}}$$

$$v_V = 3.51 \times 10^4\ \text{m/s}$$

$$v_E = \sqrt{\frac{GM_S}{r_E}}$$

$$= \sqrt{\frac{(6.67 \times 10^{-11}\ \text{N·m}^2/\text{kg}^2)(1.99 \times 10^{30}\ \text{kg})}{1.49 \times 10^{11}\ \text{m}}}$$

$$v_E = 2.98 \times 10^4\ \text{m/s}$$

Venus travels at a speed of 3.51×10^4 m/s and Earth travels more slowly at a speed of 2.98×10^4 m/s around the Sun.

Practice

Understanding Concepts

Refer to Appendix C for required data.

1. Why does the Moon, which is attracted by gravity toward Earth, not fall into Earth?
2. Why does the gravitational force on a space probe in a circular orbit around a planet not change the speed of the probe?
3. A satellite is in circular orbit 525 km above the surface of Earth. Determine the satellite's (a) speed and (b) period of revolution.
4. A satellite can travel in a circular orbit very close to the Moon's surface because there is no air resistance. Determine the speed of such a satellite, assuming the orbital radius is equal to the Moon's radius.

Applying Inquiry Skills

5. (a) Write a proportionality statement indicating the relationship between the speed of a natural or artificial satellite around a central body and the radius of the satellite's orbit.
   (b) Sketch a graph of that relationship.

Making Connections

6. Space junk is becoming a greater problem as more human-made objects are abandoned in their orbits around Earth. Research this problem using the Internet or other appropriate publications, and write a brief summary of what you discover.

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Answers

3. (a) 7.60×10^3 m/s
   (b) 5.71×10^3 s (or 1.59 h)
4. 1.68×10^3 m/s

Kepler's Laws of Planetary Motion

Centuries before telescopes were invented, astronomers made detailed observations of the night sky and discovered impressive and detailed mathematical relationships. Prior to the seventeenth century, scientists continued to believe that Earth was at or very near the centre of the universe, with the Sun and the other known planets (Mercury, Venus, Mars, Jupiter, and Saturn) travelling in orbits around Earth. Using Earth as the frame of reference, the "geocentric model" of the universe was explained by introducing complicated motions (**Figure 2**).

The detailed observations and analysis needed to invent these complex orbits were amazingly accurate and allowed scientists to predict such celestial events as solar and lunar eclipses. However, the causes of the motions were poorly understood. Then in 1543, Polish astronomer Nicolas Copernicus (1473–1543) published a book in which he proposed the "heliocentric model" of the solar system in which the planets revolve around the Sun. He deduced that the planets closer to the Sun have a higher speed than those

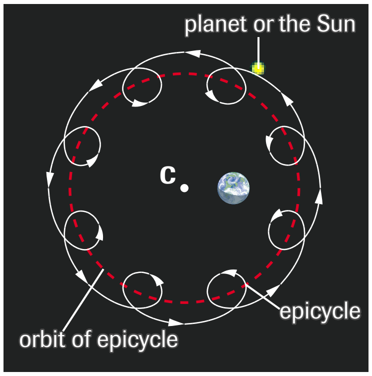

Figure 2
Using Earth as the frame of reference, the motion of the Sun and the other planets is an orbit called an *epicycle*, which itself is in an orbit around point C, located away from Earth.

farther away, which agrees with the orbital speed calculations given by $v = \sqrt{\frac{GM}{r}}$. Using the Sun's frame of reference, the motions of the planets suddenly appeared very simple (**Figure 3**).

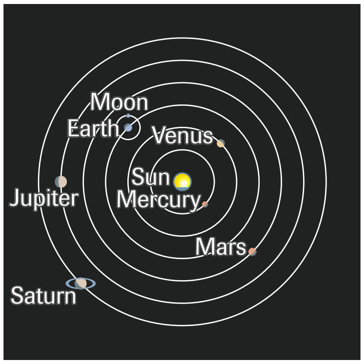

Figure 3
Using the Sun as the frame of reference, the motions of the planets were modelled as simple circles around the Sun, and the Moon was modelled as a circle around Earth.

Although Copernicus was at the forefront of the scientific revolution, his explanation of the orbits of the planets did not account for slight irregularities observed over long periods. The orbits were not exactly circles. More analysis was needed to find the true shapes of the orbits.

The next influential astronomer was Danish astronomer Tycho Brahe (1546–1601), usually called Tycho. He was hired in Denmark as a "court astronomer" to the king. For 20 years, he carried out countless naked-eye observations using unusually large instruments (**Figure 4**), accumulating the most complete and accurate observations yet made. However, after annoying those around him, he lost the king's support and in 1597, he moved from Denmark to Prague. There he spent the last years of his life analyzing his data. In 1600, shortly before his death, he hired a brilliant young mathematician to assist in the analysis. That mathematician was Johannes Kepler (1571–1630).

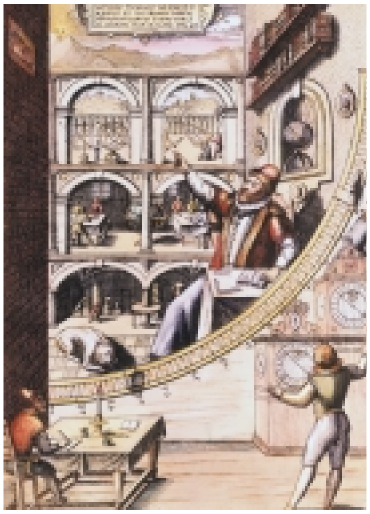

Figure 4
This large instrument, called a *quadrant*, was so precise that Tycho could measure the angular position of a star to the closest $\frac{1}{1000}$ of a degree.

Kepler, who was born and educated in Germany, moved to Prague and spent much of the next 25 years painstakingly analyzing Tycho's great volume of planetary motion data. His objective was to find the orbital shape of the motions of the planets that best fit the data. Working mainly with the orbit of Mars, for which Tycho's records were most complete, Kepler finally discovered that the only shape that fit all the data was the ellipse. He then developed three related conclusions to explain the true orbits of planets. (We now know that these conclusions also apply to the motion of any body orbiting another body, such as the Moon or a satellite orbiting Earth.) These three relationships are called *Kepler's laws of planetary motion*.

Kepler's First Law of Planetary Motion

Each planet moves around the Sun in an orbit that is an ellipse, with the Sun at one focus of the ellipse.

Figure 5 illustrates Kepler's first law. Although this law states correctly that the planetary orbits are ellipses, for most of the planets the ellipses are not very elongated. In fact, if you were to draw a scale diagram of the orbits of the planets (except for Mercury and Pluto), they would look much like circles. For example, the distance from Earth to the Sun varies by only about 3% during its annual motion about the Sun.

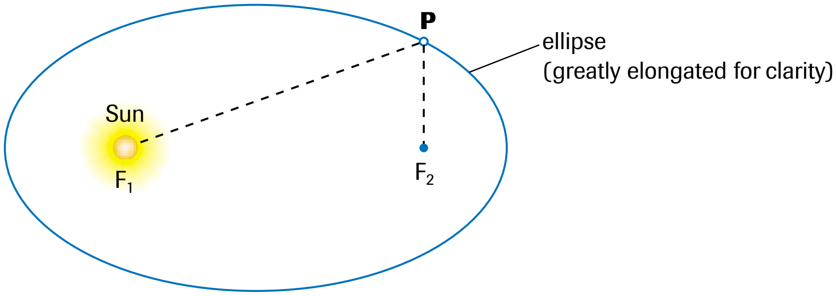

Figure 5
The orbit of a planet is an ellipse with the Sun at one focus. Based on the definition of an ellipse, for any point P, the distance $PF_1 + PF_2$ is a constant.

DID YOU KNOW?

Tycho's Problems

Tycho Brahe had personal problems, many of which were of his own doing. He was arrogant, conceited, and often quarrelsome. When he was only 19, he fought a foolish duel in which part of his nose was cut off. He had to wear a fake metal insert for the rest of his life.

Even before Kepler had established that the orbit of Mars is an ellipse, he had determined that Mars speeds up as it approaches the Sun and slows down as it moves away. *Kepler's second law of planetary motion* states the relationship precisely:

Kepler's Second Law of Planetary Motion

The straight line joining a planet and the Sun sweeps out equal areas in space in equal intervals of time.

Figure 6 illustrates Kepler's second law. The statement that equal areas are swept out in equal time intervals is equivalent to saying that each planet moves most rapidly when closest to the Sun and least rapidly when farthest from the Sun. Earth is closest to the Sun around January 4, and so is farthest from the Sun around July 5.

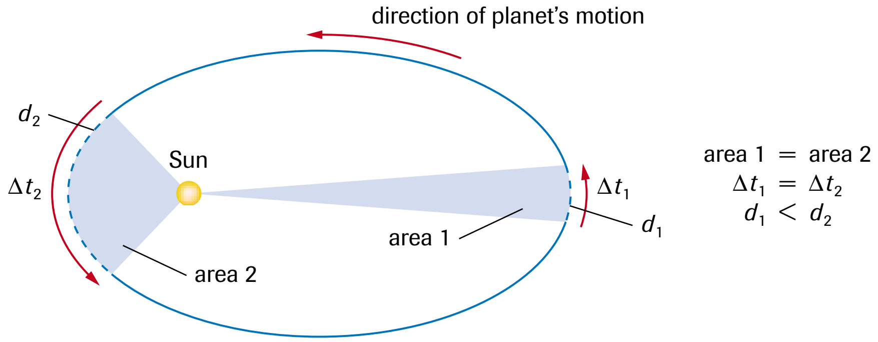

Figure 6
This ellipse is elongated to better illustrate the idea of equal areas swept out in equal time intervals.

Kepler's third law of planetary motion gives the relationship between the period T of a planet's orbit (i.e., the time taken for each revolution about the Sun) and the average distance r from the Sun:

Kepler's Third Law of Planetary Motion

The cube of the average radius r of a planet's orbit is directly proportional to the square of the period T of the planet's orbit.

Writing Kepler's third law mathematically, we have:

$$r^3 \propto T^2$$

$$r^3 = C_S T^2$$

or $$C_S = \frac{r^3}{T^2}$$

where C_S is the constant of proportionality for the Sun for Kepler's third law. In SI units, C_S is stated in metres cubed per seconds squared (m^3/s^2).

DID YOU KNOW?

Earth's Changing Seasons

Our seasons occur because Earth's axis of rotation is at an angle of about 23.5° to the plane of Earth's orbit around the Sun. As a result, the North Pole faces somewhat away from the Sun during the months close to December, giving rise to winter in the Northern Hemisphere. At the same time, the South Pole faces somewhat toward the Sun, producing summer in the Southern Hemisphere. In months close to June, the North Pole points slightly toward the Sun and the South Pole points away from it, reversing the seasons.

SAMPLE problem 2

The average radius of orbit of Earth about the Sun is 1.495×10^8 km. The period of revolution is 365.26 days.

(a) Determine the constant C_S to four significant digits.

(b) An asteroid has a period of revolution around the Sun of 8.1×10^7 s. What is the average radius of its orbit?

Solution

(a) $r_E = 1.495 \times 10^8 \text{ km} = 1.495 \times 10^{11} \text{ m}$

$T_E = 365.26 \text{ days} = 3.156 \times 10^7 \text{ s}$

$C_S = ?$

$$C_S = \frac{r^3}{T_E^2}$$

$$= \frac{(1.495 \times 10^{11}\ \text{m})^3}{(3.156 \times 10^7\ \text{s})^2}$$

$$C_S = 3.355 \times 10^{18}\ \text{m}^3/\text{s}^2$$

The Sun's constant is $3.355 \times 10^{18}\ \text{m}^3/\text{s}^2$.

(b) We can apply the Sun's constant found in (a) to this situation.

$C_S = 3.355 \times 10^{18}\ \text{m}^3/\text{s}^2$

$T = 8.1 \times 10^7\ \text{s}$

$r = ?$

$$\frac{r^3}{T^2} = C_S$$

$$r = \sqrt[3]{C_S T^2}$$

$$= \sqrt[3]{(3.355 \times 10^{18}\ \text{m}^3/\text{s}^2)(8.1 \times 10^7\ \text{s})^2}$$

$$r = 2.8 \times 10^{11}\ \text{m}$$

The average radius of the asteroid's orbit is 2.8×10^{11} m.

Kepler's findings were highly controversial because they contradicted the geocentric model of the solar system supported by the Roman Catholic Church. Indeed, in 1616, the Church issued a decree labelling the heliocentric hypothesis as "false and absurd." (This decree was made largely because Galileo supported the heliocentric hypothesis.)

Kepler's third law equation is all the more amazing when we observe that many years later, the same relationship could be obtained by applying Newton's law of universal gravitation to the circular motion of one celestial body travelling around another. We begin by equating the magnitude of the gravitational force to the product of the mass and the centripetal acceleration for a planet moving around the Sun: $\frac{GM_S m_{\text{planet}}}{r^2} = \frac{m_{\text{planet}} v^2}{r}$.

From which $v = \sqrt{\frac{GM_S}{r}}$

Since $T = \frac{2\pi r}{v}$

$$T = \frac{2\pi r}{\sqrt{\frac{GM_S}{r}}}$$

$$T^2 = \frac{4\pi^2 r^2}{\left(\frac{GM_S}{r}\right)}$$

$$= 4\pi^2 r^2 \left(\frac{r}{GM_S}\right)$$

$$T^2 = \frac{4\pi^2 r^3}{GM_S}$$

$$\frac{r^3}{T^2} = \frac{GM_S}{4\pi^2}$$

$$C_S = \frac{GM_S}{4\pi^2}$$ for the Sun, or in general

$$C = \frac{GM}{4\pi^2} = \frac{r^3}{T^2}$$

We have proven that the constant for the Sun depends only on the mass of the Sun. This relationship, however, applies to any central body about which other bodies orbit. For example, the constant for Earth C_E depends on the mass of Earth M_E and applies to the Moon or to any artificial satellite in orbit around Earth:

$$C_E = \frac{GM_E}{4\pi^2} = \frac{r_{Moon}{}^3}{T_{Moon}{}^2}$$

Today's astronomers use sophisticated Earth-bound and orbiting telescopes to gather accurate data of the motions of celestial bodies, as well as advanced computing and simulation programs to analyze the data. But astronomers will always admire the accuracy and unending hard work of the Renaissance astronomers, especially Tycho and Kepler.

LEARNING TIP

More about Kepler's Third-Law Constant

The constant of proportionality C is defined in this text as the ratio of r^3 to T^2, which is equal to the ratio $\frac{GM}{4\pi^2}$ and is measured in metres cubed per second squared. The constant could also be written as the ratio of T^2 to r^3, or $\frac{4\pi^2}{GM}$ and is measured in seconds squared per metre cubed. This latter case is found in some references.

Practice

Understanding Concepts

7. If the solar system were considered to be an isolated system, which model (geocentric or heliocentric) is the noninertial frame of reference? Explain your answer.

8. Why did Tycho not gather any data from the planets beyond Saturn?

9. Between March 21 and September 21, there are three days more than between September 21 and March 21. These two dates are the spring and fall equinoxes when the days and nights are of equal length. Between the equinoxes, Earth moves 180° around its orbit with respect to the Sun. Using Kepler's laws, explain how you can determine the part of the year during which the Earth is closer to the Sun.

10. Using the planetary data in Appendix C, calculate the ratio $\frac{r^3}{T^2}$ for each planet, and verify Kepler's third law by confirming that $r^3 \propto T^2$.

11. (a) What is the average value (in SI base units) of the constant of proportionality in $r^3 \propto T^2$ that you found in question 10?
(b) Use your answer in (a) to determine the mass of the Sun.

12. (a) Use the data of the Moon's motion (refer to Appendix C) to determine Kepler's third-law constant C_E to three significant digits for objects orbiting Earth.
(b) If a satellite is to have a circular orbit about Earth ($m_E = 5.98 \times 10^{24}$ kg) with a period of 4.0 h, how far, in kilometres, above the centre of Earth must it be? What must be its speed?

Applying Inquiry Skills

13. Go back to the ellipses you drew in the Try This Activity at the beginning of Chapter 6 and label one focus on each ellipse the "Sun." As accurately as possible, draw diagrams to illustrate Kepler's second law of planetary motion. Verify that a planet travels faster when it is closer to the Sun. (Your diagram for each ellipse will resemble **Figure 6**; you can use approximate distances along the arcs to compare the speeds.)

Making Connections

14. Astronomers have announced newly discovered solar systems far beyond our solar system. To determine the mass of a distant star, they analyze the motion of a planet around that star.
(a) Derive an equation for the mass of a central body, around which another body revolves in an orbit of known period and average radius.
(b) If a planet in a distant solar system cannot be observed directly, its effect on the central star might be observed and used to determine the radius of the planet's orbit. Describe how this is possible for a "main-sequence star" whose mass can be estimated by its luminosity. (Assume there is a single large-mass planet in orbit around the star and that the star has an observable wobble.)

Answers

11. (a) 3.36×10^{18} m^3/s^2
(b) 1.99×10^{30} kg

12. (a) 1.02×10^{13} m^3/s^2
(b) 1.3×10^4 km; 5.6×10^3 m/s

14. (a) $M = \frac{4\pi^2 r^3}{GT^2}$

SUMMARY Orbits and Kepler's Laws

- The orbits of planets are most easily approximated as circles even though they are ellipses.
- Kepler's first law of planetary motion states that each planet moves around the Sun in an orbit that is an ellipse, with the Sun at one focus of the ellipse.
- Kepler's second law of planetary motion states that the straight line joining a planet and the Sun sweeps out equal areas in space in equal intervals of time.
- Kepler's third law of planetary motion states that the cube of the average radius r of a planet's orbit is directly proportional to the square of the period T of the planet's orbit.

Section 6.2 Questions

Understanding Concepts

Refer to Appendix C for required data.

1. Apply one of Kepler's laws to explain why we are able to observe comets close to Earth for only small time intervals compared to their orbital periods. (*Hint:* A comet's elliptical orbit is very elongated.)
2. Earth is closest to the Sun about January 4 and farthest from the Sun about July 5. Use Kepler's second law to determine on which of these dates Earth is travelling most rapidly and least rapidly.
3. A nonrotating frame of reference placed at the centre of the Sun is very nearly an inertial frame of reference. Why is it not exactly an inertial frame of reference?
4. An asteroid has a mean radius of orbit around the Sun of 4.8×10^{11} m. What is its orbital period?
5. If a small planet were discovered with an orbital period twice that of Earth, how many times farther from the Sun is this planet located?
6. A spy satellite is located one Earth radius above Earth's surface. What is its period of revolution, in hours?
7. Mars has two moons, Phobos and Deimos (Greek for "Fear" and "Panic," companions of Mars, the god of war). Deimos has a period of 30 h 18 min and an average distance from the centre of Mars of 2.3×10^4 km. The period of Phobos is 7 h 39 min. What is the average distance of Phobos from the centre of Mars?

Applying Inquiry Skills

8. Show that the SI base units of $\sqrt{\frac{GM}{r}}$ are metres per second.
9. Sketch the shape of a graph of r^3 as a function of T^2 for planets orbiting the Sun. What does the slope of the line on the graph indicate?

Making Connections

10. Galileo was the first person to see any of Jupiter's moons.
    (a) Relate this important event to the works of Tycho and Kepler by researching when Galileo first discovered that Jupiter had moons and how this discovery came to pass.
    (b) After discovering these moons, what would Galileo need to know to calculate Jupiter's mass?
    (c) Would Galileo have been able to determine Jupiter's mass when he first saw the moons, or would that calculation have had to wait for awhile? (*Hint:* Kepler's first two laws were published in 1609.)