

3.1 Uniform Circular Motion



Figure 1
In performing the spiral, the female skater undergoes circular motion.

uniform circular motion motion that occurs when an object has constant speed and constant radius

centripetal acceleration instantaneous acceleration directed toward the centre of the circle

INVESTIGATION 3.1.1

Analyzing Uniform Circular Motion (p. 152)

If you are whirling a rubber stopper, attached to a string, around your head, the tension in the string (a force you can measure) keeps the stopper travelling in a circle. How do you think the force depends on such factors as the mass of the stopper, the frequency with which you are whirling the stopper, and the distance between your hand and the stopper? You will explore these relationships in a controlled experiment at the end of this chapter.

When figure skating pairs perform the spiral, the woman travels in a circular path around the man (**Figure 1**). In doing so, she is constantly changing direction, which means she is undergoing acceleration. In this section, we will analyze the factors influencing that acceleration and relate our analysis to other applications that involve motion in a circular path.

Imagine you have attached a rubber stopper to the end of a string and are whirling the stopper around your head in a horizontal circle. If both the speed of the stopper and the radius of its path remain constant, the stopper has **uniform circular motion**. **Figure 2** shows the position and velocity vectors at various positions during the motion of the stopper. Uniform circular motion also occurs if only part of the circle (an arc) is covered.

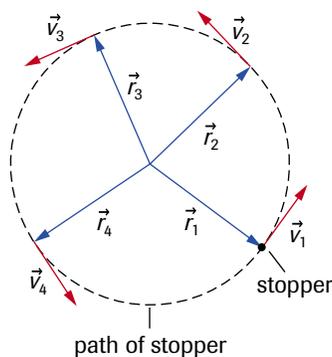


Figure 2
In uniform circular motion, the speed of the object remains constant, but the velocity vector changes because its direction changes. The radius of the path also remains constant. Notice that the instantaneous position vector (also called the radius vector) is perpendicular to the velocity vector and the velocity vectors are tangent to the circle.

Uniform circular motion occurs for the individual parts of any object spinning at a constant rate, for example, electric fans and motors, lawn mower blades, wheels (from the point of view of the centre of the wheel), and rotating rides at amusement parks. Circular or almost circular motion can also occur for objects or particles in orbits around other objects. For example, to analyze the motion of a planet orbiting the Sun, a satellite around Earth, or an electron around a nucleus, we make the assumption that the motion is uniform circular motion even though the paths are often ellipses rather than circles.

As you learned in Chapter 1, an object travelling at a constant speed is undergoing acceleration if the direction of the velocity is changing. This is certainly true for uniform circular motion. The type of acceleration that occurs in uniform circular motion is called **centripetal acceleration**.

Centripetal acceleration is an instantaneous acceleration. Investigation 3.1.1 in the Lab Activities section at the end of this chapter will help you to better understand the related concepts. In this investigation, you will perform a controlled experiment to explore the factors that affect centripetal acceleration and the force that causes it.

Practice

Understanding Concepts

- (a) What does “uniform” mean in the expression “uniform circular motion”?
(b) Give some examples of uniform circular motion, other than those in the text.
- How can a car moving at a constant speed be accelerating at the same time?

The Direction of Centripetal Acceleration

Recall that the defining equation for instantaneous acceleration is $\vec{a} = \lim_{\Delta t \rightarrow 0} \frac{\Delta \vec{v}}{\Delta t}$. To apply this equation to uniform circular motion, we draw vector diagrams and perform vector subtractions. **Figure 3** shows what happens to $\Delta \vec{v}$ as Δt decreases. As the time interval approaches zero, the direction of the change of velocity $\Delta \vec{v}$ comes closer to pointing toward the centre of the circle. From the defining equation for instantaneous acceleration, you can see that the direction of the acceleration is the same as the direction of the change of velocity. We conclude that *the direction of the centripetal acceleration is toward the centre of the circle*. Notice that the centripetal acceleration and the instantaneous velocity are perpendicular to each other.

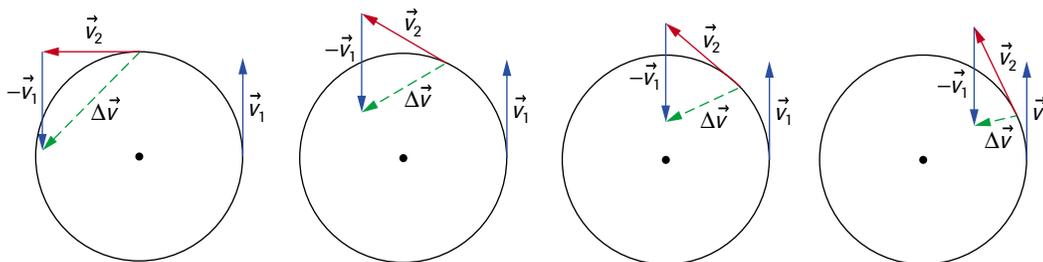


Figure 3

As the time interval between \vec{v}_1 and \vec{v}_2 is made shorter and shorter, $\Delta \vec{v}$ comes closer and closer to pointing toward the centre of the circle. In the diagram on the far right, Δt is very small and the $\Delta \vec{v}$ vector is nearly perpendicular to the instantaneous velocity vector \vec{v}_2 .

Practice

Understanding Concepts

- A sports car moves through a horizontal circular arc at a constant speed.
 - What is the direction of the instantaneous acceleration at every point in the arc?
 - Draw a sketch showing the directions of the instantaneous velocity and instantaneous acceleration at two different positions.
- If the direction of an object moving with uniform circular motion is reversed, what happens to the direction of the centripetal acceleration?

The Magnitude of Centripetal Acceleration

We can derive an equation for the magnitude of the centripetal acceleration in terms of the instantaneous speed and the radius of the circle. **Figure 4(a)** shows a particle in uniform circular motion as it moves from an initial position \vec{r}_1 to a subsequent position \vec{r}_2 ; its corresponding velocities are \vec{v}_1 and \vec{v}_2 . Since we have uniform circular motion, $|\vec{v}_1| = |\vec{v}_2|$. The change in position is $\Delta \vec{r}$ and the change in velocity is $\Delta \vec{v}$. Both of these quantities involve a vector subtraction as shown in **Figures 4(b)** and (c). Both triangles are isosceles because $|\vec{r}_1| = |\vec{r}_2|$ and $|\vec{v}_1| = |\vec{v}_2|$. Since $\vec{v}_1 \perp \vec{r}_1$ and $\vec{v}_2 \perp \vec{r}_2$, the two triangles are similar. Therefore, the following equation can be written:

$$\frac{|\Delta \vec{v}|}{|\vec{v}|} = \frac{|\Delta \vec{r}|}{|\vec{r}|} \quad \text{where } |\vec{v}| = |\vec{v}_1| = |\vec{v}_2| \text{ and } |\vec{r}| = |\vec{r}_1| = |\vec{r}_2|$$

$$\text{or } |\Delta \vec{v}| = \frac{|\vec{v}| \times |\Delta \vec{r}|}{|\vec{r}|}$$

Now, the magnitude of the centripetal acceleration \vec{a}_c is

$$|\vec{a}_c| = \lim_{\Delta t \rightarrow 0} \frac{|\Delta \vec{v}|}{\Delta t}$$

DID YOU KNOW?

Understanding “Centripetal”

The word “centripetal” was coined by Sir Isaac Newton, from the Latin *centrum* (“centre”) and *petere* (“to seek”). Do not confuse centripetal, or “centre-seeking,” forces with centrifugal forces. “Fugal” means to flee, so centrifugal forces are “centre-fleeing” forces.

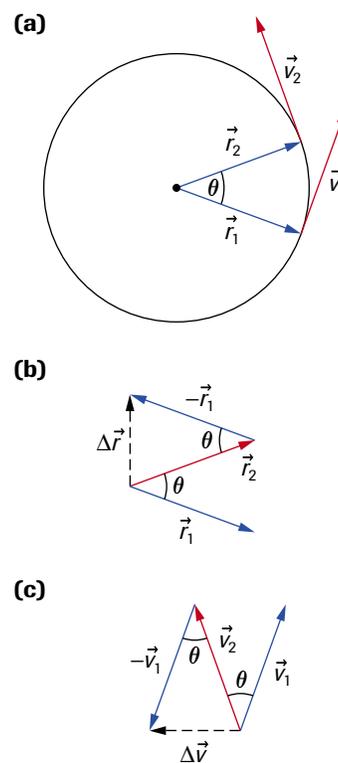


Figure 4

Derivation of the equation for the magnitude of the centripetal acceleration

- The position and velocity vectors
- The change of position
- The change of velocity

LEARNING TIP

The Position Vector

In Chapter 1, we used the symbols \vec{d} for position and $\Delta\vec{d}$ for change of position (displacement). Here we use the symbols \vec{r} and $\Delta\vec{r}$ for the corresponding quantities, as a reminder that our position vector is the radius vector, equal in magnitude to the radius of the circle.

LEARNING TIP

Two Different Accelerations

Remember that centripetal acceleration is an instantaneous acceleration. Thus, even if the magnitude of the velocity is changing for an object in circular motion (i.e., the circular motion is not “uniform” since the speed is either increasing or decreasing), the magnitude of the centripetal acceleration can still be found using the equation

$$a_c = \frac{v^2}{r}.$$

However, there is also another acceleration, called the *tangential acceleration*, that is parallel to the velocity (i.e., tangential to the circle). This acceleration changes only the velocity's magnitude. The centripetal acceleration, which is perpendicular to the velocity, changes only the velocity's direction.

We can divide both sides of the $|\Delta\vec{v}|$ equation by Δt to obtain

$$\frac{|\Delta\vec{v}|}{\Delta t} = \frac{|\vec{v}|}{|\vec{r}|} \times \frac{|\Delta\vec{r}|}{\Delta t}$$

$$\text{Therefore, } |\vec{a}_c| = \lim_{\Delta t \rightarrow 0} \left(\frac{|\vec{v}|}{|\vec{r}|} \times \frac{|\Delta\vec{r}|}{\Delta t} \right)$$

Now the magnitude of the instantaneous velocity is

$$|\vec{v}| = \lim_{\Delta t \rightarrow 0} \frac{|\Delta\vec{r}|}{\Delta t}$$

$$\text{Therefore, } |\vec{a}_c| = \frac{|\vec{v}|}{|\vec{r}|} \times |\vec{v}|$$

Hence, the magnitude of the centripetal acceleration is

$$a_c = \frac{v^2}{r}$$

where a_c is the centripetal acceleration, v is the speed of the object undergoing uniform circular motion, and r is the radius of the circle or arc.

The equation $a_c = \frac{v^2}{r}$ makes sense because as the speed of an object in circular motion increases at a constant radius, the direction of the velocity changes more quickly requiring a larger acceleration; as the radius becomes larger (at a constant speed), the direction changes more slowly, meaning a smaller acceleration.

▶ SAMPLE problem 1

A child on a merry-go-round is 4.4 m from the centre of the ride, travelling at a constant speed of 1.8 m/s. Determine the magnitude of the child's centripetal acceleration.

Solution

$$v = 1.8 \text{ m/s}$$

$$r = 4.4 \text{ m}$$

$$a_c = ?$$

$$\begin{aligned} a_c &= \frac{v^2}{r} \\ &= \frac{(1.8 \text{ m/s})^2}{4.4 \text{ m}} \\ a_c &= 0.74 \text{ m/s}^2 \end{aligned}$$

The child's centripetal acceleration has a magnitude of 0.74 m/s^2 .

For objects undergoing uniform circular motion, often the speed is not known, but the radius and the period (the time for one complete trip around the circle) are known. To determine the centripetal acceleration from this information, we know that the speed is constant and equals the distance travelled ($2\pi r$) divided by the period of revolution (T):

$$v = \frac{2\pi r}{T}$$

Substituting this expression into the centripetal acceleration equation $a_c = \frac{v^2}{r}$, we obtain

$$a_c = \frac{4\pi^2 r}{T^2}$$

For high rates of revolution, it is common to state the frequency rather than period. The frequency f is the number of revolutions per second, or the reciprocal of the period. It is measured in cycles per second or hertz (Hz), but may be stated mathematically as s^{-1} . Since $f = \frac{1}{T}$, the equation for centripetal acceleration can be written as:

$$a_c = 4\pi^2 r f^2$$

We now have three equations for determining the magnitude of the centripetal acceleration:

$$a_c = \frac{v^2}{r} = \frac{4\pi^2 r}{T^2} = 4\pi^2 r f^2$$

Remember that the direction of the centripetal acceleration is always toward the centre of the circle.

▶ SAMPLE problem 2

Find the magnitude and direction of the centripetal acceleration of a piece of lettuce on the inside of a rotating salad spinner. The spinner has a diameter of 19.4 cm and is rotating at 780 rpm (revolutions per minute). The rotation is clockwise as viewed from above. At the instant of inspection, the lettuce is moving eastward.

Solution

$$f = (780 \text{ rev/min})(1 \text{ min}/60 \text{ s}) = 13 \text{ Hz} = 13 \text{ s}^{-1}$$

$$r = \frac{19.4 \text{ cm}}{2} = 9.7 \text{ cm} = 9.7 \times 10^{-2} \text{ m}$$

$$\vec{a}_c = ?$$

$$a_c = 4\pi^2 r f^2$$

$$= 4\pi^2 (9.7 \times 10^{-2} \text{ m})(13 \text{ s}^{-1})^2$$

$$a_c = 6.5 \times 10^2 \text{ m/s}^2$$

Figure 5 shows that since the lettuce is moving eastward, the direction of its centripetal acceleration must be southward (i.e., toward the centre of the circle). The centripetal acceleration is thus $6.5 \times 10^2 \text{ m/s}^2$ [S].

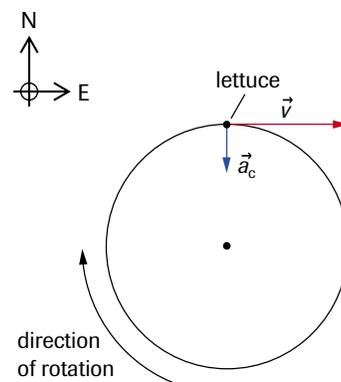


Figure 5
Piece of lettuce undergoing uniform circular motion for Sample Problem 2

▶ SAMPLE problem 3

Determine the frequency and period of rotation of an electric fan if a spot at the end of one fan blade is 15 cm from the centre and has a centripetal acceleration of magnitude $2.37 \times 10^3 \text{ m/s}^2$.

Solution

$$a_c = 2.37 \times 10^3 \text{ m/s}^2 \quad f = ?$$

$$r = 15 \text{ cm} = 0.15 \text{ m} \quad T = ?$$

DID YOU KNOW?

Changing Technologies

The technology of recorded music continually changes. A record turntable rotating at a constant rate of $33\frac{1}{3}$ revolutions per minute (rpm) has a stylus that moves faster through the groove near the outside edge of the record than near the middle. Thus, the sound information must be compressed more closely as the groove gets closer to the centre. In compact discs, however, the laser pickup moves at a constant speed across the information tracks. Since the information bits are evenly spaced, the disc must rotate at a lower frequency as the laser moves from the centre toward the outside.

Answers

- (b) $2.0 \times 10^1 \text{ m/s}^2$
- 0.10 m/s^2
- (a) $8.97 \times 10^{22} \text{ m/s}^2$
(b) $1.2 \times 10^2 \text{ m/s}^2$
(c) 1.6 m/s^2
- 5.5 m/s
- $7.6 \times 10^6 \text{ s}$ or 88 d

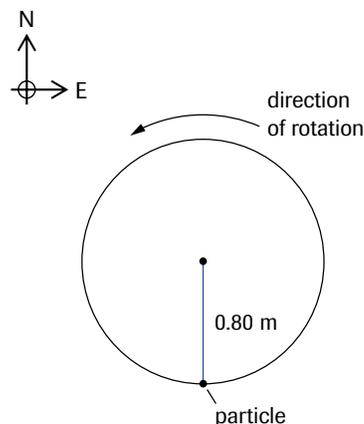


Figure 6
For question 5

$$a_c = 4\pi^2 r f^2$$

$$f^2 = \frac{a_c}{4\pi^2 r}$$

$$f = \pm \sqrt{\frac{a_c}{4\pi^2 r}}$$

$$= \pm \sqrt{\frac{2.37 \times 10^3 \text{ m/s}^2}{4\pi^2 (0.15 \text{ m})}}$$

$$= \pm 2.0 \times 10^1 \text{ s}^{-1}$$

$$f = 2.0 \times 10^1 \text{ Hz} \quad (\text{rejecting a negative frequency as meaningless})$$

The frequency of rotation of the electric fan is $2.0 \times 10^1 \text{ Hz}$.

$$T = \frac{1}{f}$$

$$= \frac{1}{2.0 \times 10^1 \text{ Hz}}$$

$$= \frac{1}{2.0 \times 10^1 \text{ s}^{-1}}$$

$$T = 5.0 \times 10^{-2} \text{ s}$$

The period of rotation of the fan is $5.0 \times 10^{-2} \text{ s}$.

Practice

Understanding Concepts

- Figure 6** shows a particle undergoing uniform circular motion at a speed of 4.0 m/s .
(a) State the direction of the velocity vector, the acceleration vector, and the radius vector at the instant shown.
(b) Calculate the magnitude of the centripetal acceleration.
- You are whirling a ball on the end of a string in a horizontal circle around your head. What is the effect on the magnitude of the centripetal acceleration of the ball if
(a) the speed of the ball remains constant, but the radius of the circle doubles?
(b) the radius of the circle remains constant, but the speed doubles?
- At a distance of 25 km from the eye of a hurricane, the wind is moving at 180 km/h in a circle. What is the magnitude of the centripetal acceleration, in metres per second squared, of the particles that make up the wind?
- Calculate the magnitude of the centripetal acceleration in the following situations:
(a) An electron is moving around a nucleus with a speed of $2.18 \times 10^6 \text{ m/s}$. The diameter of the electron's orbit is $1.06 \times 10^{-10} \text{ m}$.
(b) A cowhand is about to lasso a calf with a rope that is undergoing uniform circular motion. The time for one complete revolution of the rope is 1.2 s . The end of the rope is 4.3 m from the centre of the circle.
(c) A coin is placed flat on a vinyl record, turning at $33\frac{1}{3} \text{ rpm}$. The coin is 13 cm from the centre of the record.
- A ball on a string, moving in a horizontal circle of radius 2.0 m , undergoes a centripetal acceleration of magnitude 15 m/s^2 . What is the speed of the ball?
- Mercury orbits the Sun in an approximately circular path, at an average distance of $5.79 \times 10^{10} \text{ m}$, with a centripetal acceleration of magnitude $4.0 \times 10^{-2} \text{ m/s}^2$. What is its period of revolution around the Sun, in seconds? in "Earth" days?

Applying Inquiry Skills

- Graph the relationship between the magnitude of centripetal acceleration and
(a) the speed of an object in uniform circular motion (with a constant radius)
(b) the radius of a circle (at a constant speed)
(c) the radius of rotation (at a constant frequency)

SUMMARY**Uniform Circular Motion**

- Uniform circular motion is motion at a constant speed in a circle or part of a circle with a constant radius.
- Centripetal acceleration is the acceleration toward the centre of the circular path of an object travelling in a circle or part of a circle.
- Vector subtractions of position and velocity vectors can be used to derive the equations for centripetal acceleration.

► **Section 3.1 Questions**

Understanding Concepts

- Describe examples of uniform circular motion that can occur
 - in a kitchen
 - in a repair workshop
 - in situations in which there is less than a complete circle
- Two balls at the ends of two strings are moving at the same speed in horizontal circular paths. One string is three times as long as the other. Compare the magnitudes of the two centripetal accelerations.
- Calculate the magnitude of the centripetal acceleration in each of the following situations:
 - A satellite is travelling at 7.77×10^3 m/s in a circular orbit of radius 6.57×10^6 m from the centre of Earth.
 - A motorcycle is racing at 25 m/s on a track with a constant radius of curvature of 1.2×10^2 m.
- What is the magnitude of centripetal acceleration due to the daily rotation of an object at Earth's equator? The equatorial radius is 6.38×10^6 m.
 - How does this acceleration affect a person's weight at the equator?
- Patrons on an amusement park ride called the Rotor stand with their backs against the wall of a rotating cylinder while the floor drops away beneath them. To keep from sliding downward, they require a centripetal acceleration in excess of about 25 m/s^2 . The Rotor has a diameter of 5.0 m. What is the minimum frequency for its rotation? (The vertical force required to support the rider's weight is supplied by static friction with the wall.)
- A car, travelling at 25 m/s around a circular curve, has a centripetal acceleration of magnitude 8.3 m/s^2 . What is the radius of the curve?
- The Moon, which revolves around Earth with a period of about 27.3 d in a nearly circular orbit, has a centripetal acceleration of magnitude $2.7 \times 10^{-3} \text{ m/s}^2$. What is the average distance from Earth to the Moon?

Applying Inquiry Skills

- You are asked to design a controlled experiment to verify the mathematical relationships between centripetal acceleration and other relevant variables.
 - Describe how you would conduct the experiment.
 - What are the most likely sources of random and systematic error in your experiment? How does your experiment help keep these sources of error within reasonable bounds?

Making Connections

- A biophysicist seeks to separate subcellular particles with an analytic ultracentrifuge. The biophysicist must determine the magnitude of the centripetal acceleration provided by the centrifuge at various speeds and radii.
 - Calculate the magnitude of the centripetal acceleration at 8.4 cm from the centre of the centrifuge when it is spinning at 6.0×10^4 rpm. Express your answer in terms of g .
 - What are some other uses of centrifuges?