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Session 4: Uncertainties & Mathematical Requirements

Diploma Programme

Session 4: Success Criteria

After this session, participants will be able to:

- **Understand expectations around uncertainties and the propagation of error**
- **Recognize mathematical requirements for students**
- **Appreciate the role of significant figures in uncertainties**
- **Compare different approaches for graphical uncertainty**

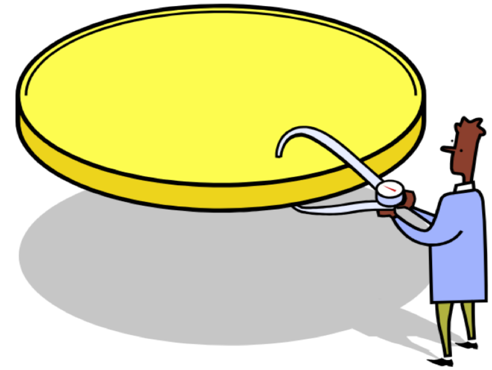
Dealing with Uncertainties

$$x_{\text{Best}} \pm \Delta x_{\text{Uncertainty}}$$

UNCERTAINTIES

"Unless you can measure what you are speaking about and express it in numbers, you have scarcely advanced to the stage of science."—*Lord Kelvin*

**Physics is a quantitative science—
it deals with numbers and measurements.
If we are to understand the knowledge of physics we must be aware of the nature of measurement.**



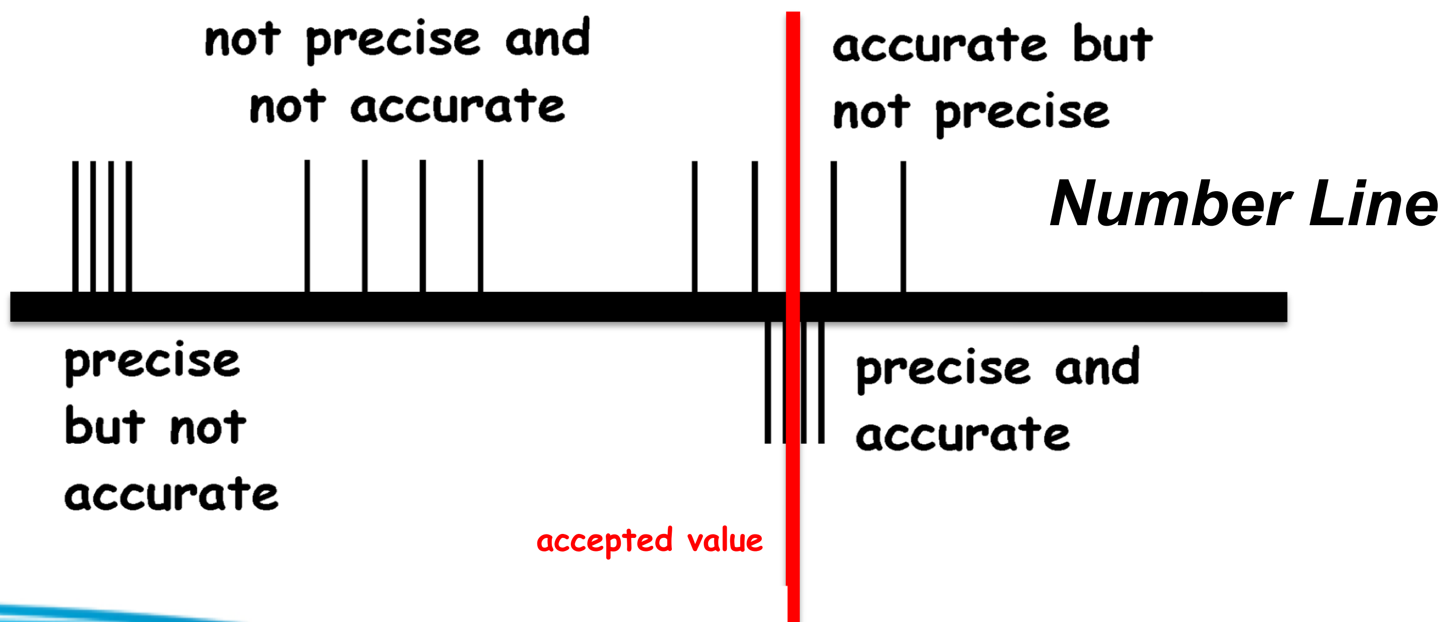
IB Physics Data Booklet, First Exam 2016, Subtopic 1.2

$$\text{If } y = a \pm b \text{ then } \Delta y = \Delta a + \Delta b$$

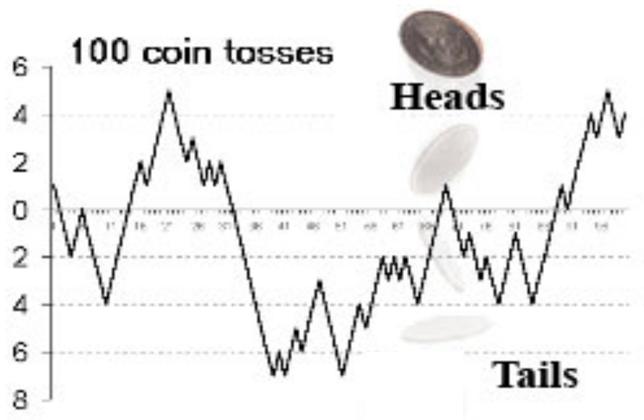
$$\text{If } y = \frac{ab}{c} \text{ then } \frac{\Delta y}{y} = \frac{\Delta a}{a} + \frac{\Delta b}{b} + \frac{\Delta c}{c}$$

$$\text{If } y = a^n \text{ then } \frac{\Delta y}{y} = \left| n \frac{\Delta a}{a} \right|$$

Measurements are **ACCURATE** if the systematic error is small, and **PRECISE** if the random error is small.



COUNTING NUMBERS



HEADS	TAILS	TOSSES
44	56	100

$$N_{\text{Heads}} \pm \Delta N = N_{\text{Heads}} \pm \sqrt{N}$$

$$N_{\text{Heads}} \pm \Delta N = 44 \pm \sqrt{44} = 44 \pm 7$$

COUNTING NUMBERS

If an average of 15.3 radioactive events occurred in one minute when measured over twenty minutes, then average count and its uncertainty are:

$$N \pm \Delta N = 15.3 \pm \sqrt{15} \approx 15 \pm 4 \text{ counts per minute}$$

UNCERTAINTIES

When we **add** or **subtract** quantities, we **add their raw or absolute uncertainties.**

$$L = r + w \quad \rightarrow \quad L \pm \Delta L$$

$$r \pm \Delta r = (6.1 \pm 0.1)\,\text{cm}$$

$$w \pm \Delta w = (12.6 \pm 0.2)\,\text{cm}$$

$$L = 6.1\,\text{cm} + 12.6\,\text{cm} = 18.7\,\text{cm}$$

$$\Delta L = \Delta r + \Delta w = 0.1\,\text{cm} + 0.2\,\text{cm} = 0.3\,\text{cm}$$

$$\therefore L \pm \Delta L = (18.7 \pm 0.3)\,\text{cm}$$

UNCERTAINTIES

When we find the **product** or **quotient** of two or more quantities, we **add the percentages of uncertainties.**

$$A = L \times W = 24.3\,\text{cm} \times 11.8\,\text{cm} = 286.74\,\text{cm}^2$$

$$\Delta L = \Delta W = \pm 0.1\,\text{cm} \qquad \Delta A\% = \Delta L\% + \Delta W\%$$

$$\Delta A\% = \left(\frac{0.1}{24.3} \times 100\right) + \left(\frac{0.1}{11.8} \times 100\right) = 0.411\% + 0.847\%$$

$$\Delta A\% = 1.258\% \approx 1\%$$

$$A \pm \Delta A = 286.74\,\text{cm}^2 \pm 2.8674\,\text{cm}^2$$

$$\therefore A \pm \Delta A \approx (287 \pm 3)\,\text{cm}^2$$

UNCERTAINTIES

Squaring a quantity is the same as multiplying it by itself. We double the percentage of uncertainty. When cubing a quantity we triple the percentage of uncertainty.

$$y = x^3 \quad \text{where} \quad x \pm \Delta x = (4.3 \pm 0.2)\,\text{m} \qquad x^3 = 79.507\,\text{m}^3$$

$$\Delta y\% = 3(\Delta \text{x}\%) = 3 \times \frac{0.2}{4.3} \times 100 = 13.9534\%$$

$$y \pm \Delta y\% \approx 80\,\text{m}^3 \pm 14\%$$

$$\Delta y = 13.9534\% \times 79.507\,\text{m}^3 = \pm 11.09393\,\text{m}^3$$

$$y \pm \Delta y = 79.507\,\text{m}^3 \pm 11.09393\,\text{m}^3$$

$$\therefore\ y \pm \Delta y \approx (80. \pm 11)\,\text{m}^3 \approx (8 \pm 1) \times 10\,\text{m}^3$$

UNCERTAINTIES

Similarly, for the **square root,** for n = 0.5 (or any power):

$$y = \sqrt{x} = x^{\frac{1}{2}} \quad \text{where} \quad x \pm \Delta x = \left(4.3 \pm 0.2\right)\text{cm}^2$$

$$\Delta y\% = \frac{1}{2}\left(\Delta \text{x}\%\right) = \frac{1}{2} \times \frac{0.2}{4.3} \times 100 = 2.325\%$$

$$y \pm \Delta y = 2.0736\,\text{cm} \pm 0.0482\ \text{cm}$$

$$\therefore\ y \pm \Delta y \approx \left(2.07 \pm 0.05\right)\ \text{cm}$$

NO CALCULATION CAN IMPROVE PRECISION

The result of addition and/or subtraction should be rounded off so that it has the same number of decimal places (to the right of the decimal point) as the quantity in the calculation having the least number of decimal places.

Adding & Subtracting

A sum or difference is not more precise than the least precise number.

$$262\,\text{mA} + 17.8\,\text{mA} = 279.8\,\text{mA} \approx 280.\text{mA}$$

280 mA understood

$$3.1\ \text{cm} - 0.57\ \text{cm} = 2.53\ \text{cm} \approx 2.5\ \text{cm}$$

NO CALCULATION CAN IMPROVE PRECISION

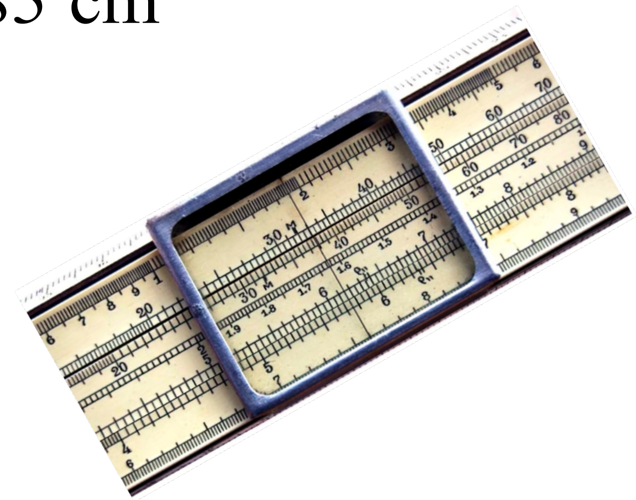
The result of multiplication and/or division should be rounded off so that it has as many significant figures as the least precise quantity used in the calculation.

Multiplying & Dividing

A product or quotient has no more significant digits than the number with the least number of significant digits.

$$13.3\ \text{cm} \times 6.4\ \text{cm} = 85.12\ \text{cm}^2 \approx 85\ \text{cm}^2$$

$$\frac{13.6}{6.4} = 2.125 \approx 2.1$$



RULES OF THE MEASURE

General Rule For Stating Uncertainties:

Experimental uncertainties should be rounded to one or two significant figures.

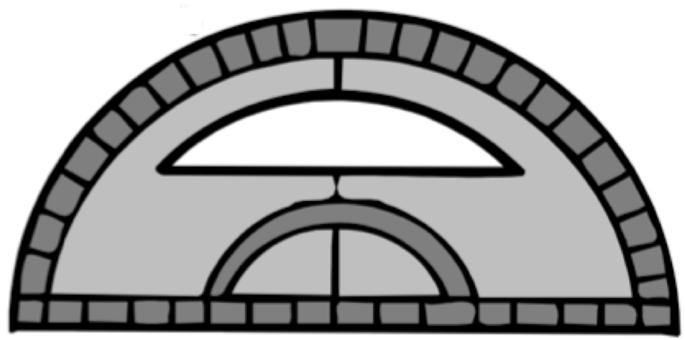
General Rule for Stating Answers:

The least significant figure in any stated answer should be of the same order of magnitude (in the same decimal position) as the uncertainty.

$$g \pm \Delta g = (9.81734 \pm 0.0217)\,\mathrm{m\,s^{-2}}$$

$$\therefore g \pm \Delta g \approx (9.82 \pm 0.02)\,\mathrm{m\,s^{-2}}$$

The ‘rules’ are general guidelines only and there are exceptions.
See “The Physics Teacher” Vol 51, Sept. 2013, 340-343.



Correct Form	Incorrect Form
32.2° ± 0.1°	32.2° ± 1°
(32.2 ± 0.2)°	(32.2 ± 1.5)°
(32 ± 1)°	(32 ± 0.5)°

ACTIVITY 7:

Now try answering some multiple choice exam questions dealing with uncertainties.

Multiple Choice Exam Questions on Uncertainties

A or B or C or D?

Question 1

The power dissipated in a resistor of resistance ***R*** carrying a current ***I*** is equal to $I^2 R$. The value of ***I*** has an uncertainty of ±2% and the value of *R* has an uncertainty of ±10%. The value of the uncertainty in the calculated power dissipation is

A. ±8%

B. ±12%

C. ±14%

D. ±20%

Multiple Choice Exam Questions on Uncertainties

Question 1

The power dissipated in a resistor of resistance ***R*** carrying a current ***I*** is equal to $I^2 R$. The value of ***I*** has an uncertainty of ±2% and the value of *R* has an uncertainty of ±10%. The value of the uncertainty in the calculated power dissipation is

A. ±8%

B. ±12%

C. ±14%

D. ±20%

$$P = I^2 R$$

$$I @ 2\% \rightarrow I^2 @ 4\%$$

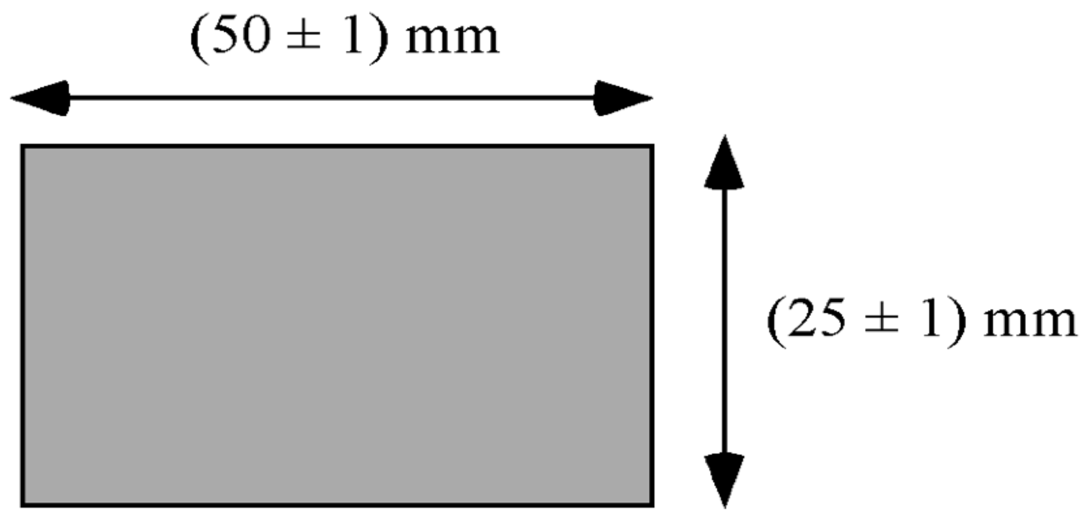
$$R @ 10\%$$

$$P @ 4\% + 10\% = 14\%$$

Multiple Choice Exam Questions on Uncertainties

Question 2

The lengths of the sides of a rectangular plate are measured, and the diagram shows the measured values with their uncertainties. What is the best estimate of the percentage uncertainty in the calculated **area** of the plate?



A. ±2%

B. ±4%

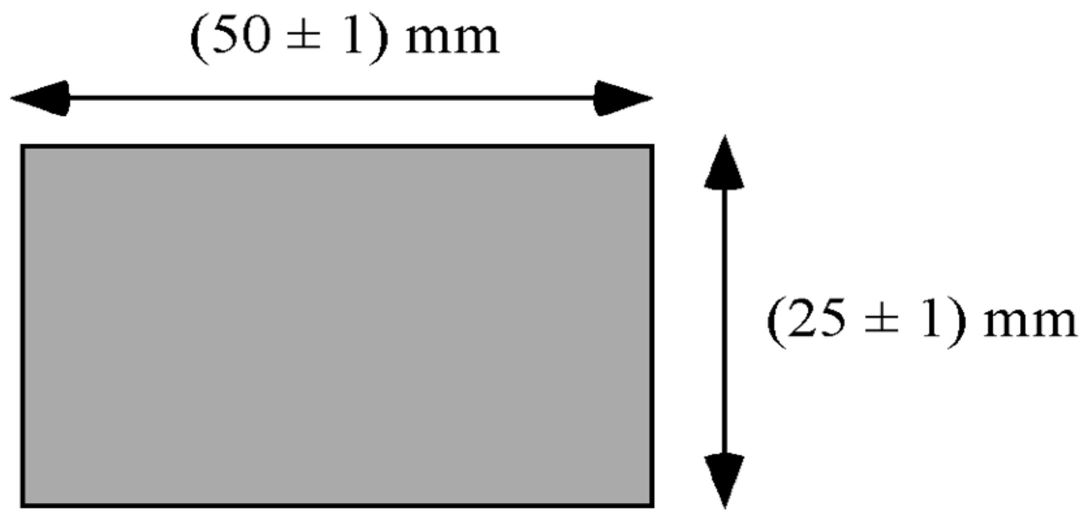
C. ±6%

D. ±8%

Multiple Choice Exam Questions on Uncertainties

Question 2

The lengths of the sides of a rectangular plate are measured, and the diagram shows the measured values with their uncertainties. What is the best estimate of the percentage uncertainty in the calculated **area** of the plate?



A. ±2%

B. ±4%

C. ±6%

D. ±8%

$$\frac{1}{50}\cdot 100 = 2\% \qquad \frac{1}{25}\cdot 100 = 4\%$$

$$\text{area}\% = 2\% + 4\% = 6\%$$

Question 3

The resultant force acting on an object is measured to an accuracy of ±4%. The mass of the object is measured to an accuracy of ±2%. The acceleration of the object can be calculated to an accuracy of approximately

A. ±2%

B. ±4%

C. ±6%

D. ±8%

Multiple Choice Exam Questions on Uncertainties

Question 3

The resultant force acting on an object is measured to an accuracy of ±4%. The mass of the object is measured to an accuracy of ±2%. The acceleration of the object can be calculated to an accuracy of approximately

A. ±2%

B. ±4%

C. ±6%

D. ±8%

$$a = \frac{F}{m} \rightarrow \frac{F @ 4\%}{m @ 2\%}$$

$$a @ 4\% + 2\% = 6\%$$

Question 4

Natalie measured the mass and speed of a glider. The percentage uncertainty in her measurement of the mass is 3% and in the measurement of the speed is 10%. Her calculated value of the kinetic energy of the glider will have an uncertainty of

A. ±30%

B. ±23%

C. ±13%

D. ±10%

Question 4

Natalie measured the mass and speed of a glider. The percentage uncertainty in her measurement of the mass is 3% and in the measurement of the speed is 10%. Her calculated value of the kinetic energy of the glider will have an uncertainty of

A. ±30%

B. ±23%

C. ±13%

D. ±10%

$$E_{\mathrm{k}} \propto mv^2$$

$$m @ 3\% \quad \text{and} \quad v @ 10\% \;\; \text{or} \;\; v^2 @ 20\%$$

$$E_{\mathrm{k}} @ 3\% + 20\% = 23\%$$

Question 5

A student measures a distance several time. The readings lie between 49.8 cm and 50.2 cm. This measurement is best recorded as

A. (49.8 ±0.2) cm

B. (49.8 ± 0.2) cm

C. (50.0 ± 0.2) cm

D. (50.0 ± 0.4) cm

Question 5

A student measures a distance several time. The readings lie between 49.8 cm and 50.2 cm. This measurement is best recorded as

A. (49.8 ±0.2) cm

B. (49.8 ± 0.2) cm

C. (50.0 ± 0.2) cm

D. (50.0 ± 0.4) cm

$$\text{ave} = \frac{(49.8 + 50.2)\,\text{cm}}{2} = 50.0\,\text{cm}$$

$$\text{uncert} = \pm\frac{\text{range}}{2} = \pm\frac{(50.2 - 49.8)\,\text{cm}}{2}$$

$$\text{uncert} = \pm\frac{0.4\text{cm}}{2} = \pm 0.2\,\text{cm}$$

Multiple Choice Exam Questions on Uncertainties

Question 6

The area *A* of a circle is equal to πr^2. If the the radius *r* of a circle is measured to be 0.10 m, what is the percentage of uncertainty in the area?

A. 1%

B. 2%

C. 5%

D. 20%

Multiple Choice Exam Questions on Uncertainties

Question 6

The area *A* of a circle is equal to π r^2. If the the radius *r* of a circle is measured to be 0.10 m, what is the percentage of uncertainty in the area?

A. 1%

B. 2%

C. 5%

D. 20%

$r \pm \Delta r$ = (0.10 ± 0.01) m Δr = least count

There is no uncertainty in π.

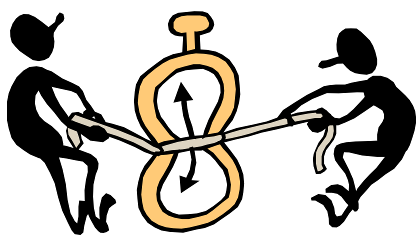
ΔA% = twice percentage of *r*

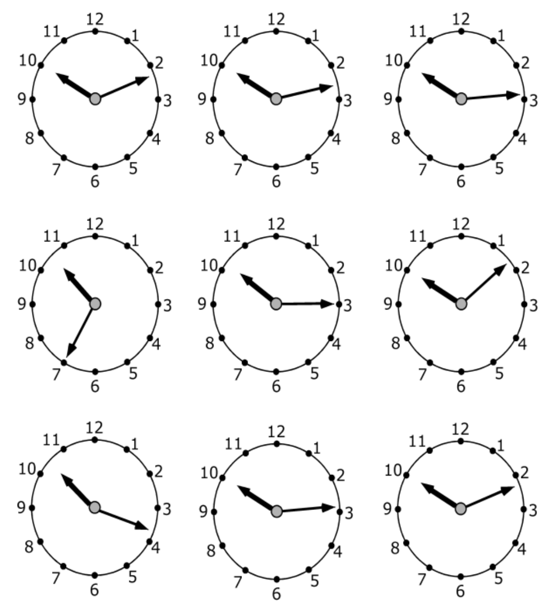
ΔA% = 2 x (0.01/0.10) x 100%

2 x 10% = 20% uncertainty in area

ACTIVITY 8: Determining the average time and the uncertainty from a number of clock images.

EXERCISE What Time Is It.pdf





End

Physics Mathematical Requirements

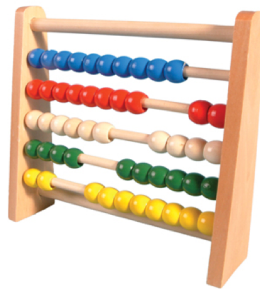
All physics students should be able to:

perform the basic arithmetic functions: addition, subtraction, multiplication and division

carry out calculations involving means, decimals, fractions, percentages, ratios, approximations and reciprocals

use standard notation (for example, 3.6×10^6)

use direct and inverse proportion



Physics Mathematical Requirements

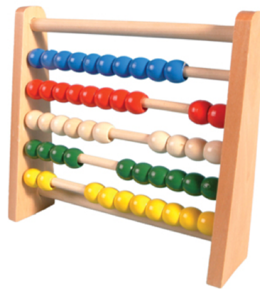
solve simple algebraic equations

solve linear simultaneous equations

plot graphs (with suitable scales and axes) including two variables that show linear and non-linear relationships

interpret graphs, including the significance of gradients, changes in gradients, intercepts and areas

draw lines (either curves or linear) of best fit on a scatter plot graph



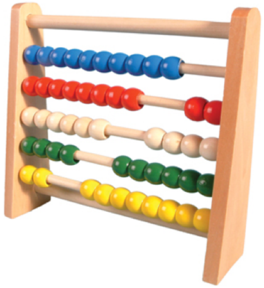
Physics Mathematical Requirements

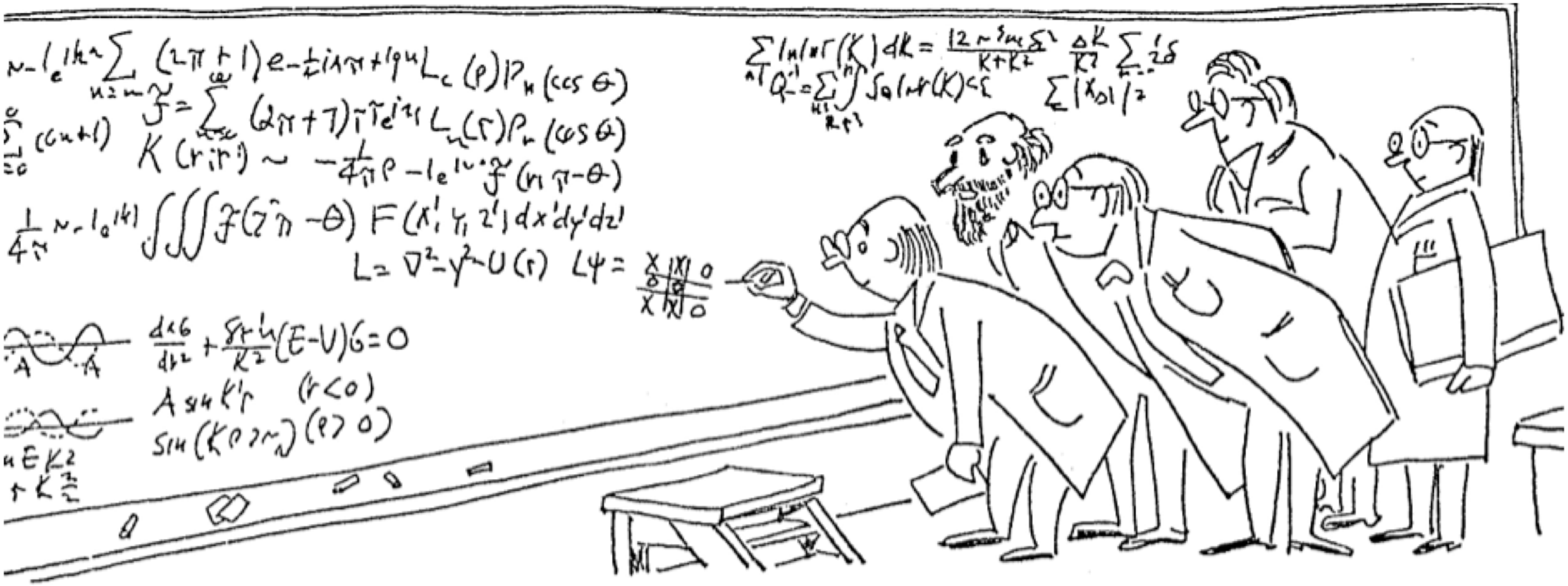
on a best-fit linear graph, construct linear lines of maximum and minimum gradients with relative accuracy (by eye) taking into account all uncertainty bars

interpret data presented in various forms (for example, bar charts, histograms and pie charts)

represent arithmetic mean using x-bar notation

express uncertainties to one or two significant figures, with justification.





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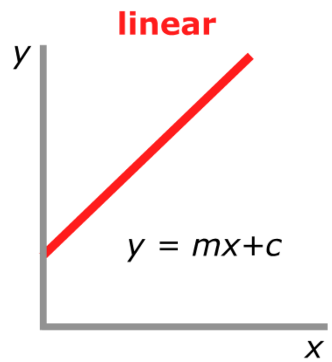
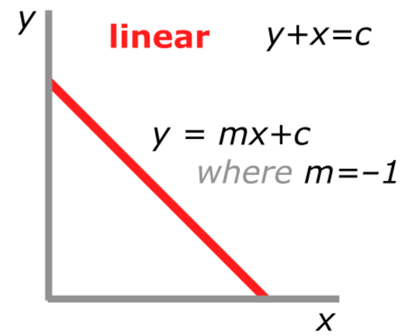
Linear Relationship

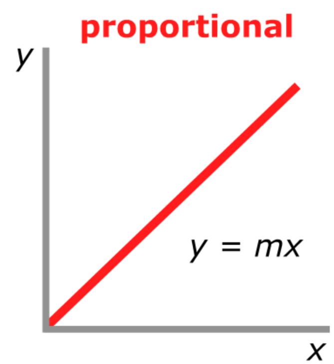
$$y = mx + c$$





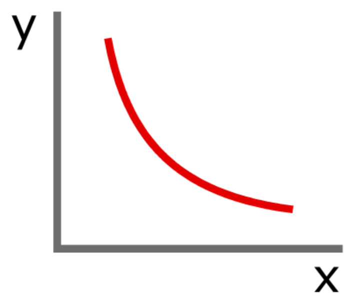
Directly Proportional

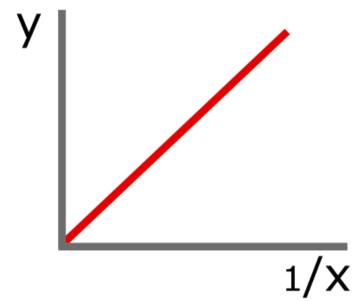
$$y = mx$$



Inversely Proportional

$$y = \frac{m}{x}$$





Solving an Algebraic Equation

$$h = \frac{1}{2} g t^2$$

$$2h = gt^2$$

$$t^2 = \frac{2h}{g}$$

$$\therefore t = \sqrt{\frac{2h}{g}}$$

Solving Linear Simultaneous Equations

$$x + y = 24 \qquad 2x - y = -6$$

$$y = 24 - x$$

$$2x - (24 - x) = -6$$

$$2x - 24 + x = -6$$

$$3x = 18$$

$$\therefore x = 6$$

Mean & Uncertainty Calculation

Assume you have calculated the free-fall value of gravity a number of times and obtained the following results:

Calculation	g / m s^{-2}
1	9.643
2	9.752
3	9.981
4	9.808
5	9.785
6	9.732

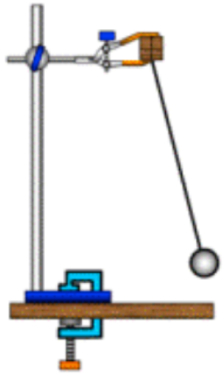
$$\Delta g_n = \pm 0.004\,\mathrm{m\,s^{-2}}$$

The average or **mean** is:

$$\bar{g} = \frac{\sum g_i}{n} = \frac{g_1 + g_2 + \ldots g_n}{n}$$

$$\bar{g} = \frac{(9.643 + 9.752 + 9.981 + 9.808 + 9.785 + 9.732)\,\mathrm{m\,s^{-2}}}{6}$$

$$\bar{g} = 9.784\,\mathrm{m\,s^{-2}}$$



Mean & Uncertainty Calculation

The simplest method to find the uncertainty in the mean is to **divide the range by 2.**

$$\text{Range} = g_{\text{maximum}} - g_{\text{minimum}} = (9.981 - 9.643)\,\text{m}\,\text{s}^{-2} = 0.338\,\text{m}\,\text{s}^{-2}$$

$$\Delta\overline{g} = \pm\frac{R}{2} = \pm\frac{0.338\,\text{m}\,\text{s}^{-2}}{2} = 0.169\,\text{m}\,\text{s}^{-2}$$

Therefore the mean and its uncertainty are:

$$\overline{g} \pm \Delta\overline{g} = (9.784 \pm 0.169)\,\text{m}\,\text{s}^{-2} \approx (9.8 \pm 0.2)\,\text{m}\,\text{s}^{-2}$$

Mean & Uncertainty Calculation

Repeated measurements using one-half the range method to determine the uncertainty do not improve the quality of the results. Standard deviation would help focus the uncertainty range. For IB physics we can approximate a statistical approach by **dividing the range by the number of measurements**. This works up to about *n* = 12.

$$\text{Range} = g_{\text{maximum}} - g_{\text{minimum}} = (9.981 - 9.643)\,\text{m}\,\text{s}^{-2} = 0.338\,\text{m}\,\text{s}^{-2}$$

$$\Delta\bar{g} = \pm\frac{R}{n} = \pm\frac{0.338\,\text{m}\,\text{s}^{-2}}{6} = 0.0563\text{m}\,\text{s}^{-2}$$

$$\bar{g} \pm \Delta\bar{g} = (9.784 \pm 0.0563)\,\text{m}\,\text{s}^{-2} \approx (9.784 \pm 0.056)\,\text{m}\,\text{s}^{-2}$$

Mean & Uncertainty Calculation

Clearly, the uncertainty has been reduced here, and we can still appreciate the mean value to three decimal places. This uncertainty tells us that the value of *g* probably lies between 9.728 and 9.840 $m\,s^{-2}$ with around a 70% change of including the true value. Hence we can accept the two-significant figure uncertainty. However, the range only gives us a statistical probability for the data, and does not detect any systematic errors and assumes a random distribution.

We can also express the uncertainty to one significant digit but that would deteriorate the quality of our results.

$$\overline{g} \pm \Delta\overline{g} = (9.784 \pm 0.0563)\,\mathrm{m\,s^{-2}} \approx (9.78 \pm 0.06)\,\mathrm{m\,s^{-2}}$$

Mean & Uncertainty Calculation

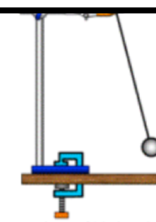
Method	Results / $m\,s^{-2}$
½ Range	9.78 ± 0.16 9.8 ± 0.2
1/*n* Range	9.784 ± 0.056 9.78 ± 0.06

Mean & Uncertainty Calculation

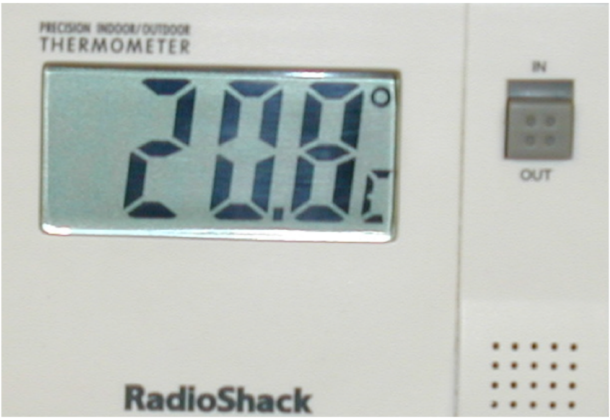
Method	Results / m s^{-2}
½ Range	9.78 ± 0.16
1/n	

$$\frac{\text{Range}}{\sqrt{n}}$$

Dividing the range by the square root of the number of measurements would be more appropriate but IB students do not need to enter the realm of statistics.



Express uncertainties to one or two significant figures, with justification.

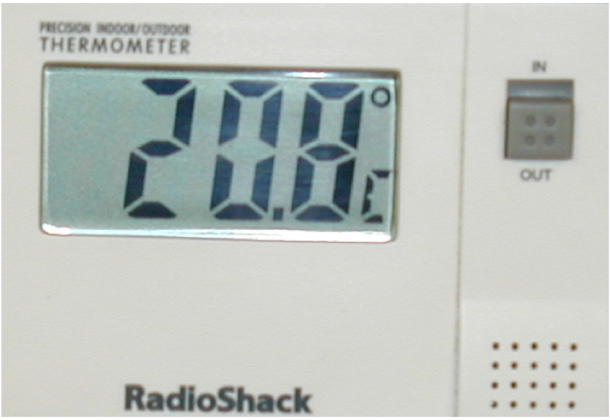


What is the temperature?

Assuming an uncertainty of the least count you might want to say that the temperature here is:

$$T \pm \Delta T = (20.8 \pm 0.1)°\text{C}$$

Express uncertainties to one or two significant figures, with justification.



Although the digital thermometer has a **resolution** of 0.1 C° the manufacture's specification sheet tells us that it has an **accuracy** of ±1 C°.

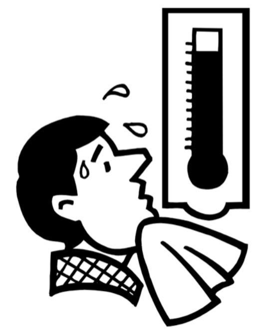
Hence, the best estimated temperature is (20.8 ± 1.0)°C.

Note that temperature is °C but a change in temperature is C°.

But isn't the temperature always 20.8°C ?

"A man with one thermometer knows what the temperature is, but a man with several thermometers is never sure.”

—Segal's Law



There are two basic sources of uncertainty in experimental measurements: random errors and equipment calibrations. The **rationale** for two-significant figures in an uncertainty is derived from a complex statistical analyses (due to the scatter of repeated measurements, random noise, calibrations issues, processing fractional or percentages of uncertainty, etc.) which are beyond the scope of high school work. However, when precision and accuracy differ we can be **justified** in stating an uncertainty to two significant figures.

Express uncertainties to one or two Significant figures, with justification.

In most high school physics investigations the precision and the uncertainty will be expressed to one significant figure only.

$$s \pm \Delta s = (42.2 \pm 0.2)\,\text{mm}$$

$$m \pm \Delta m = (467 \pm 1)\,\text{kg}$$

Express uncertainties to one or two Significant figures, with justification.

The following would be **inconsistent** and should be avoided.

$$s \pm \Delta s = (42.2 \pm 0.05)\,\mathrm{mm}$$

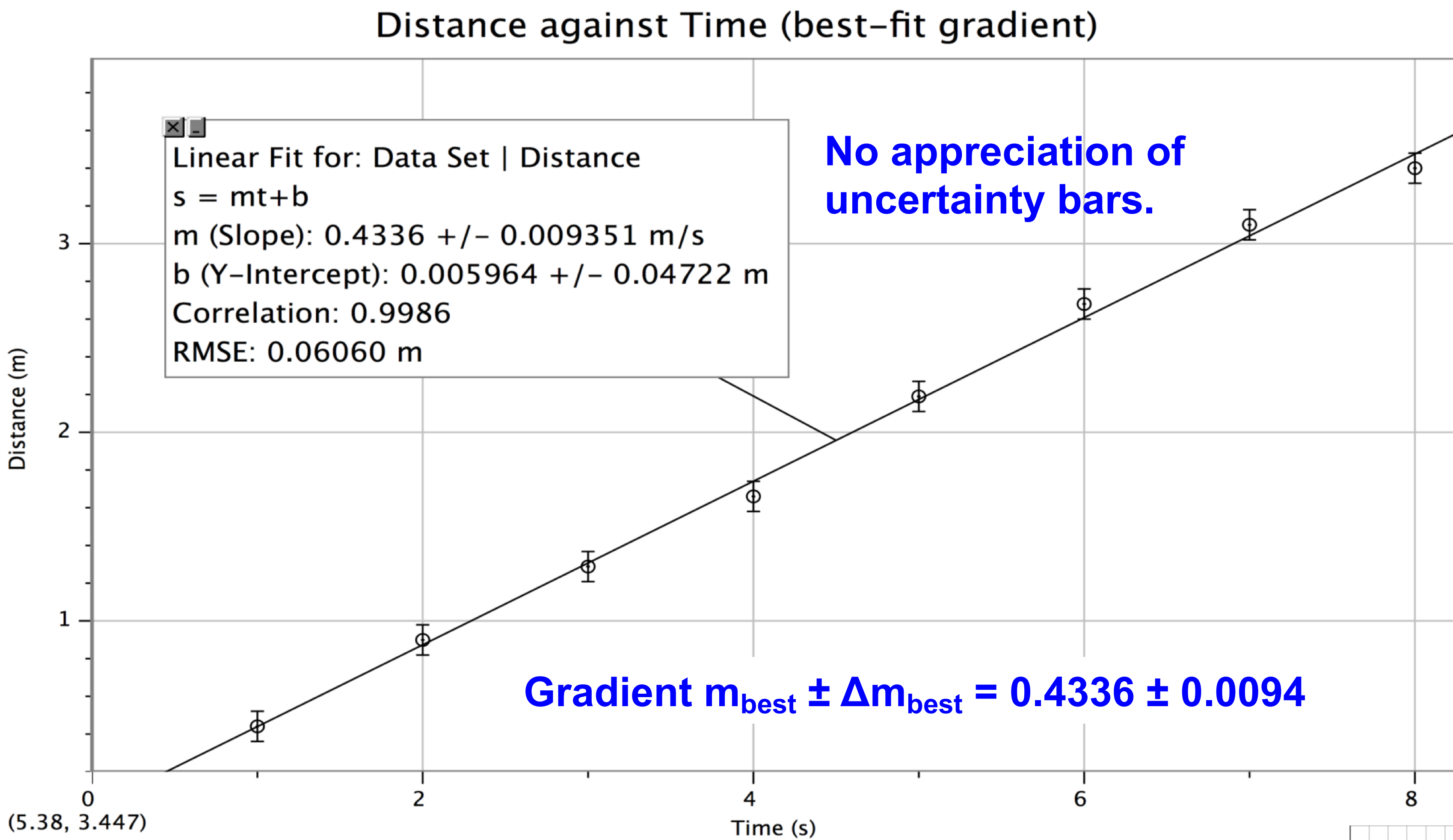
$$m \pm \Delta m = (467 \pm 0.1)\,\mathrm{kg}$$

On a best-fit linear graph, construct linear lines of maximum and minimum gradients with relative accuracy (by eye) taking into account all uncertainty bars.

First we construct the best-fit linear line on a scatter graph. This is an automatic function in most graphing and spreadsheet software programs. The software will also indicate the gradient, the *y*-intercept and the statistical uncertainty in these values.

Here is how the computer does this. A proposed straight-line is determined–a trend line for the given data based on all the data points. From each datum point to the line, a square is constructed. The computer does this over and over until the proposed linear-line and the best-fit line (with the least differences in the sum of the squares area) are one and the same. The best-fit line is also called the “least-squares” line because of the way it is calculated is like a weighted average. The actual formula is complex, but the computer does this quickly and with accuracy and precision.

Here is an example…



No appreciation of uncertainty bars.

Gradient $m_{best} \pm \Delta m_{best}$ = 0.4336 ± 0.0094

Automatic best-straight line, gradient, *y*-intercept and statistical uncertainties.

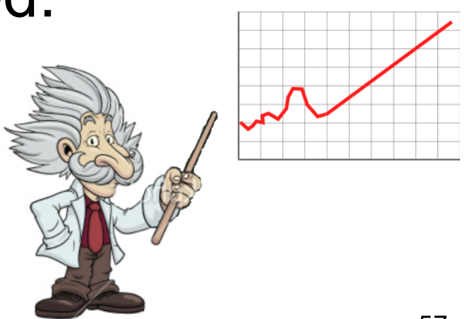
The software determines the best-fit straight line, which is a statistical determination of minimizing the absolute differences between a regression line and all of the data points.

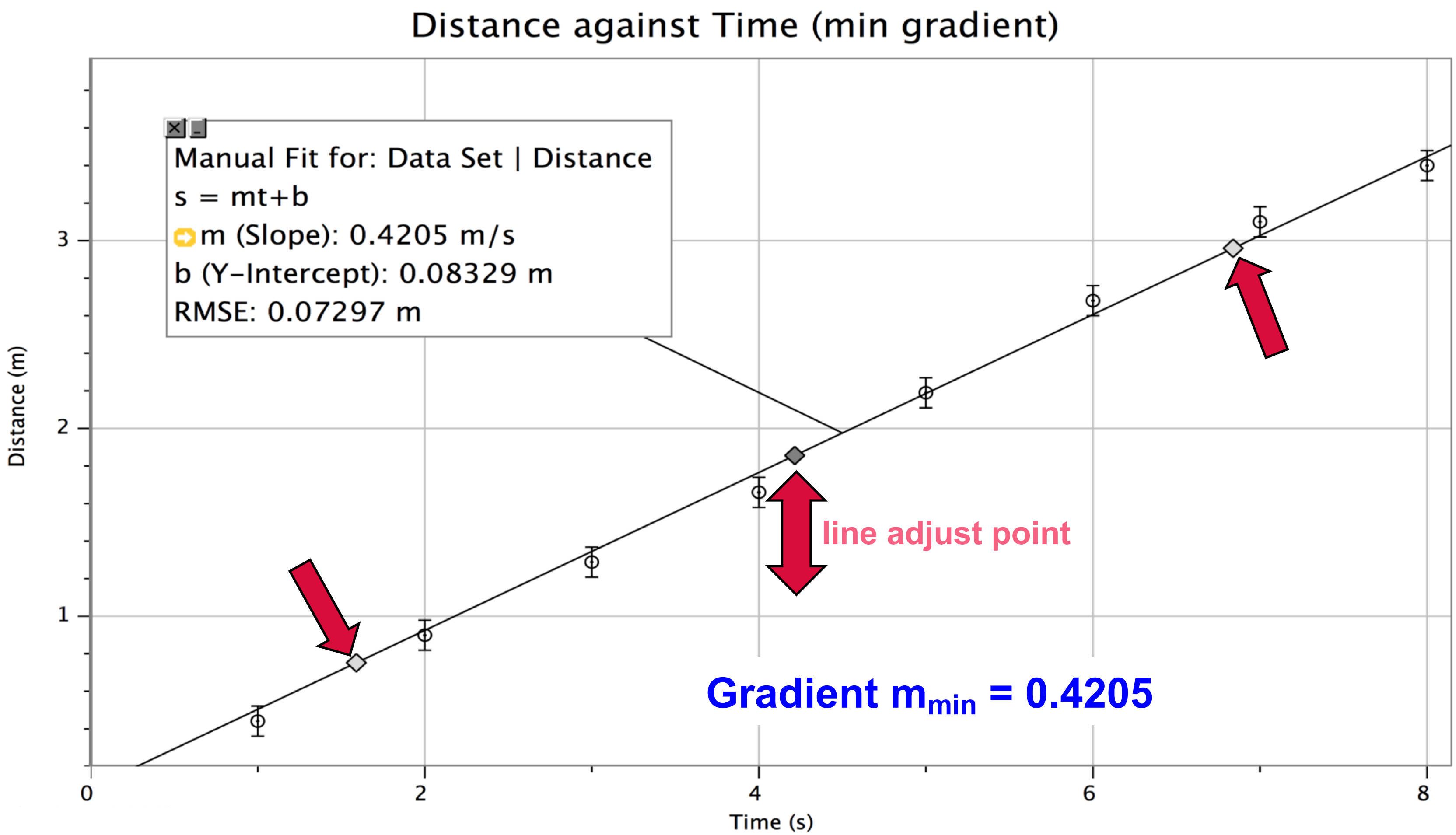
This method does not appreciate the uncertainty bars but rather is a statistical analysis of the scatter of the data points.

The **Standard Deviation** yields the uncertainty. In this case we have a gradient of 0.4336 ± 0.00931. The slope represents speed, and so we can say that the speed is (0.434 ± 0.009) m s^{-1} which is about a 2% uncertainty.

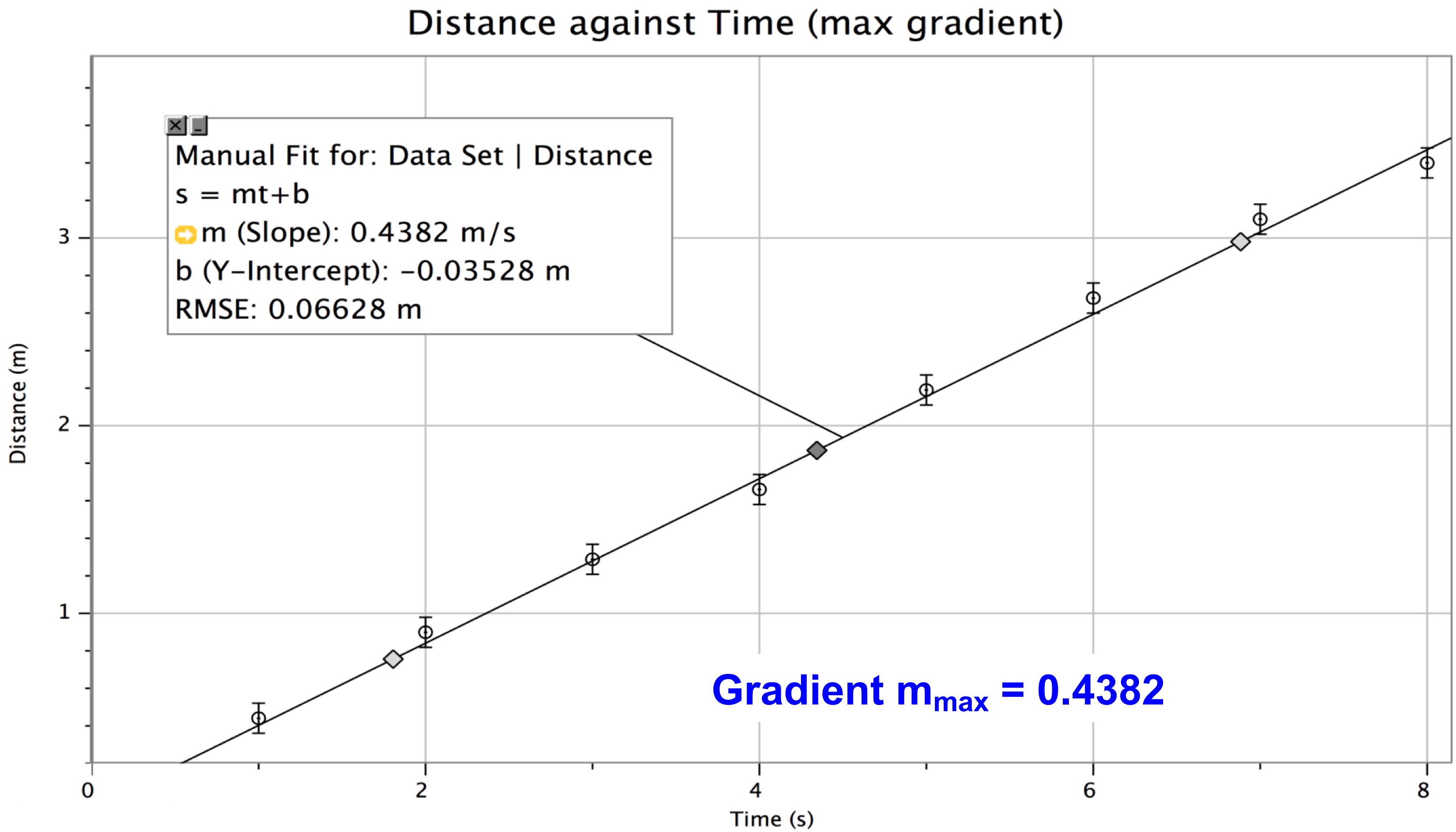
TO FIND THE UNCERTAINTY IN THE GRADIENT BY EYE USING THE UNCERTAINTY BARS we start by letting the computer determine the best-straight line and the equation for this line.

Next you insert a ‘hand-drawn’ straight line which is close to the best-straight line equation. This is called a ‘manual’ line with the form $y = m x + c$. Then, using the three indicators on the manual line, move the indicators points to adjust the line and its resulting equation. The center line-point keeps the gradient the same but moves the line up and down. The left and right indicator points adjust the slope of the straight line. You can now determine the minimum and maximum gradients using the uncertainty bars. Numerical values are displayed.





Minimum gradient using uncertainty bars, estimation by eye method.



Gradient m_{max} = 0.4382

Maximum gradient using uncertainty bars, estimation by eye method.

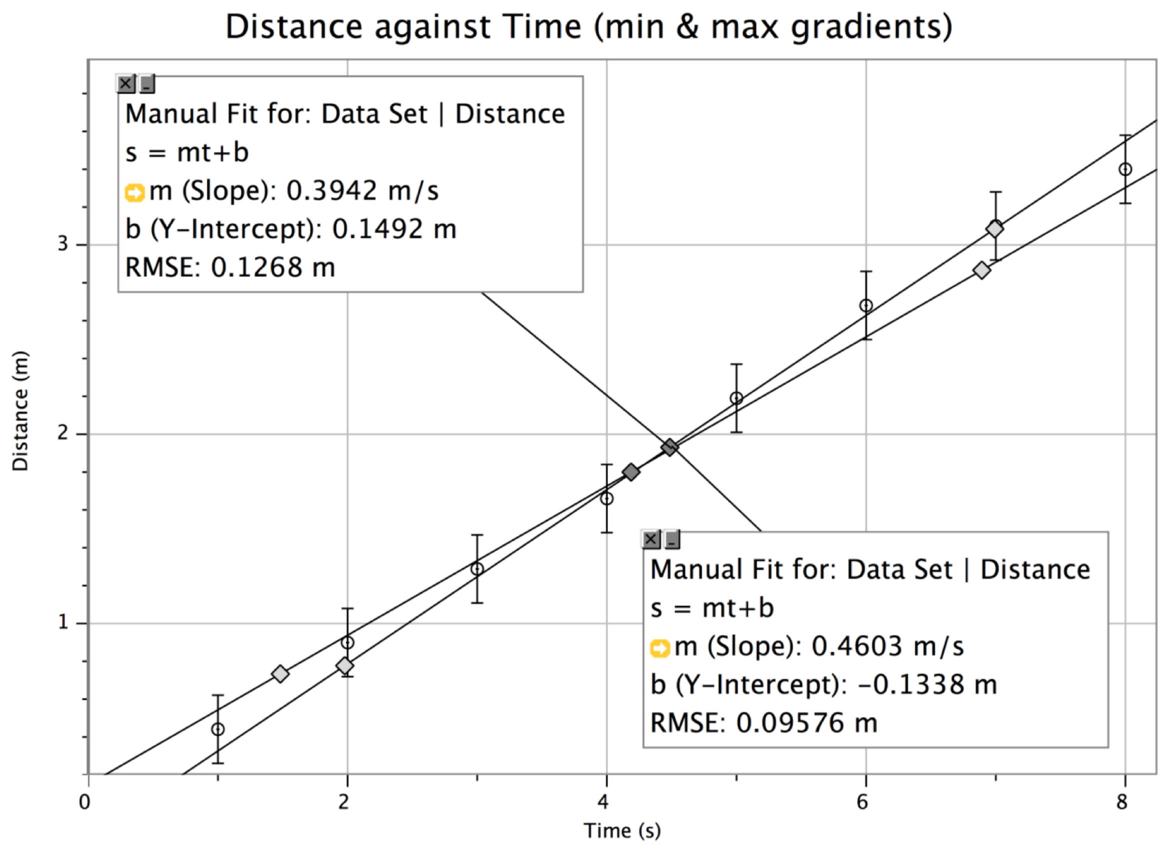
$$\Delta m_{Eye} = \pm \frac{m_{Max} - m_{Min}}{2} = \frac{0.4382 - 0.4205}{2}$$

$$\Delta m_{Eye} = \pm 0.00885 \approx \pm 0.009$$

$$m_{Best} \pm \Delta m_{Eye} \approx 0.434 \pm 0.009$$

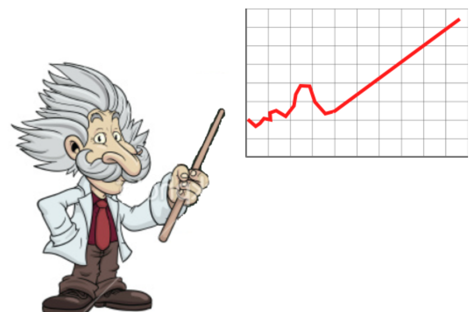
Compare this manual method of eye estimation to the software regression line uncertainty: 0.434 ± 0.009.

What do you think?



The same data but with larger uncertainty bars. The results is now about 7% compared to 2% with smaller uncertainty bars.

$$m_{Best} \pm \Delta m_{Eye} = 0.4336 \pm 0.03305 \approx 0.43 \pm 0.03$$



Students may use the automatic regression line method or the method of estimating the minimum and the maximum gradients by eye-ing up all the uncertainty bar ranges.

ACTIVITY 9: Constructing a Graph and Determining the Uncertainty in the Best Straight Line.

Using the Data from the Support File you are to plot the data points, construct uncertainty bars, determine the best-straight line and its gradient; then, using the uncertainty bars of all or most of the data points, establish the minimum and maximum gradients allowed and hence determine the uncertainty in the best straight line.

EXERCISE Gradient Problem.pdf

End