5.5 Parallel and Perpendicular Lines

GOAL

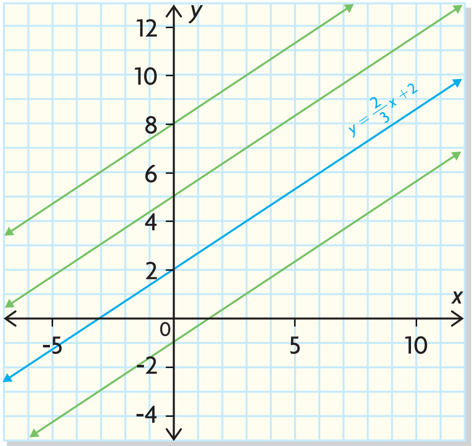
Determine and apply properties and equations of parallel and perpendicular lines.

YOU WILL NEED

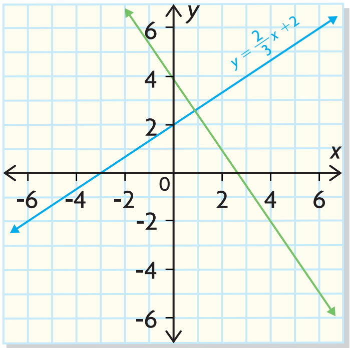
- grid paper
- graphing calculator
- protractor (optional)

INVESTIGATE the Math

The graph below could represent the rows of vines in a Niagara vineyard.



This graph could represent two jet-plane trails that intersect at right angles.



? How can you tell from the equation of a line whether it is parallel or perpendicular to a given line?

A. Graph and label each of the following lines on separate grids.

$$y = \frac{2}{3}x - 4 \qquad y = -\frac{1}{5}x + 3$$

B. Carefully draw two lines parallel to each of the original lines, and then complete the following table.

	Rise	Run	Slope	y-intercept	Equation
First Original Line					$y = \frac{2}{3}x - 4$
Parallel line 1					
Parallel line 2					
Second Original Line					$y = -\frac{1}{5}x + 3$
Parallel line 1					
Parallel line 2					

C. For each of the lines in part A, write two linear equations that represent lines parallel to the original line.

D. Draw two lines that are perpendicular to each of the original lines, and then complete the following table.

	Rise	Run	Slope	y-intercept	Equation
First Original Line					$y = \frac{2}{3}x - 4$
Perpendicular line 1					
Perpendicular line 2					
Second Original Line					$y = -\frac{1}{5}x + 3$
Perpendicular line 1					
Perpendicular line 2					

E. For each of the lines in part A, write two more linear equations that represent lines perpendicular to the original line.

Reflecting

F. How were the equations of lines parallel to $y = \frac{2}{3}x - 4$ and $y = -\frac{1}{5}x + 3$ like the original equations? How were they different?

G. How were the equations of the lines perpendicular to $y = \frac{2}{3}x - 4$ and $y = -\frac{1}{5}x + 3$ related to the equations of the original lines?

APPLY the Math

EXAMPLE 1 Reasoning about slope to determine whether lines are parallel or perpendicular

Determine which of the following lines are parallel and which are perpendicular to the line defined by $x - 2y = 4$.

a) $y = -2x + 8$ **b)** $y = \frac{2}{4}x + 2$ **c)** $y = \frac{4}{5}x + 2$

Kimmy's Solution

Original line:

$$x - 2y = 4$$
$$-2y = -x + 4$$
$$\frac{-2y}{-2} = \frac{-x + 4}{-2}$$
$$y = \frac{-x}{-2} + \frac{4}{-2}$$
$$y = \frac{1}{2}x - 2$$

I rearranged the original line into the form $y = mx + b$ so that I could determine its slope.

So, $m = \frac{1}{2}$

I determined the slope of the line in part a).

a) $y = -2x + 8$

$m = -2$

$y = -2x + 8$ is perpendicular to $y = \frac{1}{2}x + 4$

I knew that this line was perpendicular to the original line because their slopes were negative reciprocals of each other.

negative reciprocals

numbers that multiply to produce -1 are negative reciprocals of each other (e.g., $\frac{3}{4}$ and $-\frac{4}{3}$; $-\frac{1}{2}$ and 2)

b) $y = \frac{2}{4}x + 2$

I determined the slope of the line in part b).

$$m = \frac{2}{4}$$
$$= \frac{1}{2}$$

$y = \frac{2}{4}x + 2$ is parallel to $y = \frac{1}{2}x - 2$

I knew that the two lines were parallel because their slopes were equal.

c) $y = \frac{4}{5}x + 2$

$m = \frac{4}{5}$

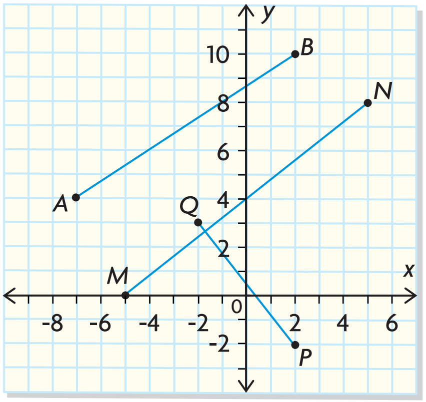
This line is neither parallel nor perpendicular to $y = \frac{1}{2}x - 2$.

I determined the slope of the line in part c).

Since the slope of this line was neither equal to nor the negative reciprocal of the slope of the original line, I knew that the lines could not be described as parallel or perpendicular.

EXAMPLE 2 Identifying perpendicularity by reasoning

Which line segments in the following diagram are perpendicular?



Liz's Solution

$\text{A}(\underset{\substack{\uparrow \\ x_1}}{-7}, \underset{\substack{\uparrow \\ y_1}}{4}) \qquad \text{B}(\underset{\substack{\uparrow \\ x_2}}{2}, \underset{\substack{\uparrow \\ y_2}}{10})$

$$\begin{aligned} m_{AB} &= \frac{10 - 4}{2 - (-7)} \\ &= \frac{6}{9} \\ &= \frac{2}{3} \end{aligned}$$

I calculated the slope of each line segment using the slope formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$.

$\text{M}(\underset{\substack{\uparrow \\ x_1}}{-5}, \underset{\substack{\uparrow \\ y_1}}{0}) \qquad \text{N}(\underset{\substack{\uparrow \\ x_2}}{5}, \underset{\substack{\uparrow \\ y_2}}{8})$

$$\begin{aligned} m_{MN} &= \frac{8 - 0}{5 - (-5)} \\ &= \frac{8}{10} \\ &= \frac{4}{5} \end{aligned}$$

$$P(\underset{\overset{\uparrow}{x_1}}{2}, \underset{\overset{\uparrow}{y_1}}{-2}) \qquad Q(\underset{\overset{\uparrow}{x_2}}{-2}, \underset{\overset{\uparrow}{y_2}}{3})$$

$$\begin{aligned} m_{PQ} &= \frac{3 - (-2)}{-2 - 2} \\ &= \frac{5}{-4} \\ &= -\frac{5}{4} \end{aligned}$$

$$\begin{aligned} m_{AB} \times m_{MN} &= \frac{2}{3} \times \frac{4}{5} \\ &= \frac{8}{15} \end{aligned}$$

I multiplied the slopes to see if any were negative reciprocals.

$$\begin{aligned} m_{AB} \times m_{PQ} &= \frac{2}{3} \times \left(-\frac{5}{4}\right) \\ &= -\frac{10}{12} \\ &= -\frac{5}{6} \end{aligned}$$

$$\begin{aligned} m_{MN} \times m_{PQ} &= \frac{4}{5} \times \left(-\frac{5}{4}\right) \\ &= -\frac{20}{20} \\ &= -1 \end{aligned}$$

I knew that the line segments $\overline{MN}$ and $\overline{PQ}$ were perpendicular because the product of their slopes was -1.

The line segments $\overline{MN}$ and $\overline{PQ}$ are perpendicular.

EXAMPLE 3 Reasoning about slope to determine the equation of a line that is parallel to another line

Determine the equation of the line that is parallel to $y = -\frac{2}{7}x + 3$ and passes through the point (14, 9).

Rahim's Solution

$$y = mx + b$$

I started with the general equation of a line.

Determine the slope:

$$m = -\frac{2}{7}$$

Since the new line is parallel to $y = -\frac{2}{7}x + 3$, I knew that it had the same slope.

$y = -\frac{2}{7}x + \text{b}$

I substituted the slope into my equation.

Determine the y-intercept:

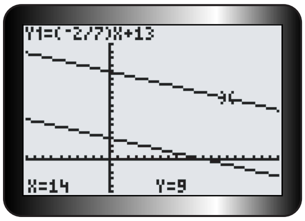
$$\begin{aligned} 9 &= -\frac{2}{7}(14) + \text{b} \\ 9 &= -4 + \text{b} \\ 9 + 4 &= -4 + 4 + \text{b} \\ 13 &= \text{b} \end{aligned}$$

Since the line passed through the point (14, 9), I used $x = 14$ and $y = 9$ in the equation of the line to determine b.

I simplified, and then used inverse operations to solve for b.

Therefore, $y = -\frac{2}{7}x + 13.$

I substituted my value for b to complete the equation.



I used a graphing calculator to check that the equation I found was parallel to the original equation and passed through (14, 9).

EXAMPLE 4 Selecting a strategy to determine the equation of a line that is perpendicular to another line

Determine the equation of the line that is perpendicular to $y = 3x + 1$ and has the same y-intercept.

Priya's Solution: Using a strategy based on slope and y-intercept

$y = \text{m}x + \text{b}$

I knew that if I could determine the line's slope and y-intercept, I could write its equation in the form $y = \text{m}x + \text{b}$.

Determine the slope:

For $y = 3x + 1$:

$\text{m} = 3$

The slope of the given line was 3 because the line was in the form $y = \text{m}x + \text{b}$.

For the new perpendicular line:

$$m = \frac{-1}{3}$$

$$= -\frac{1}{3}$$

The given line had a slope of 3, but I needed to find its negative reciprocal. I rewrote 3 as $\frac{3}{1}$. This made it easier for me to get the negative reciprocal.

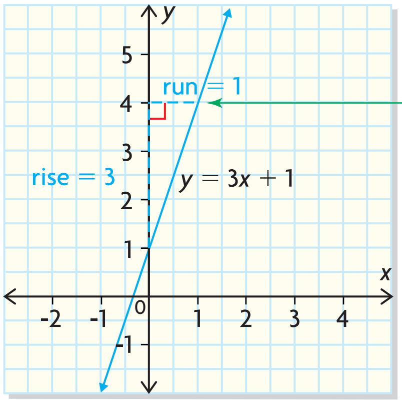
Determine the *y*-intercept:

$$b = 1$$

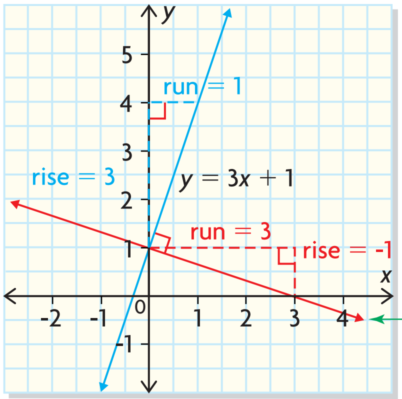
I knew that the *y*-intercept of the new line had to be the same as the *y*-intercept of the line $y = 3x + 1$.

The equation is $y = -\frac{1}{3}x + 1$.

Jacob's Solution: Reasoning from a graph



The slope was $\frac{3}{1}$ and the *y*-intercept was 1, so I used this to draw a graph of the original line.



I rotated the original line 90° to get my new line.

$$\text{new slope} = \frac{1}{-3} = -\frac{1}{3}$$

y-intercept $= 1$

The equation of the new line is

$y = -\frac{1}{3}x + 1.$

I noticed that my rise and run also rotated, so the new rise was 1 and the new run was −3. I used those values to calculate the slope. The y-intercept was the same.

In Summary

Key Ideas

- The slopes of parallel lines are equal.
- The slopes of perpendicular lines are negative reciprocals.

Need to Know

- Two numbers are negative reciprocals if they have opposite signs and their denominators and numerators are exchanged.

  For example, $\frac{-2}{3}$ and $\frac{3}{2}$ are negative reciprocals.

  So are 3 and $\frac{-1}{3}$.
- The product of two negative reciprocals is always −1.

CHECK Your Understanding

1. a) State an equation of a line parallel to $y = -\frac{3}{2}x + 9$.

b) State an equation of a line perpendicular to $y = -\frac{3}{2}x + 9$.

2. Determine which of the following lines are parallel and which are perpendicular to each other.

a) $y = -\frac{1}{3}x + 2$

b) $y = -3x + 2$

c) $y = \frac{7}{2}x - 4$

d) $y = \frac{2}{7}x - 3$

e) $y = \frac{1}{3}x + 1$

f) $y = \frac{1}{-3}x - 8$

g) $y = \frac{-3}{9}x$

h) $y = \frac{-2}{7}x - 9$

PRACTISING

3. For each pair of equations, state whether the lines are parallel, perpendicular, or neither.
K

a) $y = 2x + 5$

$y = -\frac{1}{2}x - 4$

b) $y = \frac{2}{3}x - 2$

$y = -1.5x - 6$

c) $y = \frac{3}{7}x - 4$

$y = -\frac{3}{7}x - 4$

d) $x - 4y = 2$

$2x - 8y = 3$

e) $y = -0.2x - 1$

$y = -\frac{1}{5}x + 3$

f) $x - 5y + 8 = 0$

$5x - y = 0$

4. The following sets of points define the endpoints of line segments. Determine which line segments are parallel and which line segments are perpendicular.
A(6, 5) and B(12, 3)
P(−3, −4) and Q(5, 20)
G(0, −4) and H(6, −2)
U(−5, 9) and V(−6, 12)
K(2, 4) and L(6, 16)

5. Are the lines defined by the equations $y = 4$ and $x = 3$ parallel, perpendicular, or neither? Explain.

6. a) Write the equation of a line parallel to the x-axis that passes through the point (1, 4).
b) Write the equation of a line parallel to the x-axis that passes through the point (3, −8).
c) In general, what is true about the equation of any line parallel to the x-axis?

7. a) Write the equation of a line parallel to the y-axis that passes through the point (−9, 3).
b) Write the equation of a line parallel to the y-axis that passes through the point (6, 2).
c) In general, what is true about the equation of any line parallel to the y-axis?

8. Use the given information to write the equation of each line.

A **a)** a line parallel to the line defined by $y = 3x + 5$ and passing through the point $(3, -5)$

b) a line perpendicular to the line defined by $y = 3x + 5$ and passing through the point $(3, -5)$

c) a line parallel to the line defined by $3x + 2y = 7$ with y-intercept $= 3$

d) a line perpendicular to the line defined by $2x - 3y + 18 = 0$ with the same y-intercept

9. Determine the equation of a line perpendicular to $4x - 3y - 2 = 0$ with the same y-intercept as the line defined by $3x + 4y = -12$.

10. Determine the equation of a line perpendicular to $2x - 5y = 6$ with the same x-intercept as the line defined by $3x + 8y - 15 = 0$.

11. For the given vertices, determine whether or not ΔABC is a right triangle.

a) A$(13, 3)$, B$(3, 5)$, and C$(-2, -20)$

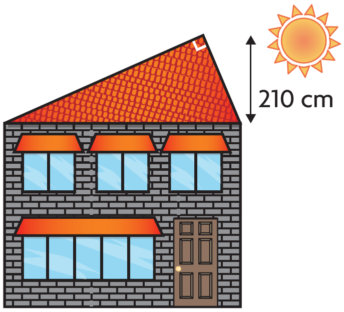
b) A$(5, 4)$, B$(-1, 2)$, and C$(2, -1)$

12. Show algebraically that the points A$(-4, 7)$, B$(6.5, 1)$, C$(-8, 0)$, and

T D$(2.5, -6)$ form a rectangle.

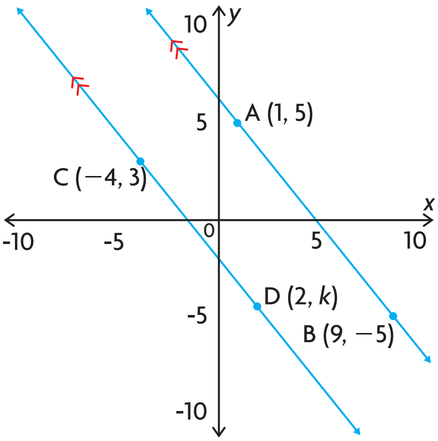
13. Show algebraically that the points E$(-9, 4)$, F$(-5, -3)$, G$(1, -4)$, and H$(-3, 3)$ form a parallelogram, but not a rectangle.

14. Mr. Rite wants his roof to be 90° at its peak and have a slope of $-\frac{7}{2}$ on the sunny side of the house. If the height of his roof must be 210 cm, how wide is his house?

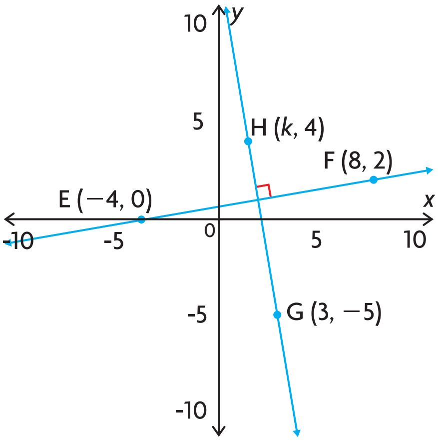


15. Determine the value of k in each graph.

a)



b)



16. Explain why a line cannot be perpendicular to $y = \frac{3}{4}x + 2$ and also be parallel to $y = \frac{4}{5}x - 8$.
C

Extending

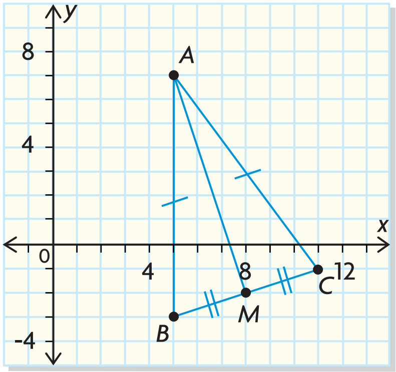
17. A line segment has endpoints A(1, −5) and B(4, 1).

a) Determine the coordinates of two points, C and D, that would make ABCD a parallelogram.

b) Determine the coordinates of two points, C and D, that would make ABCD a rectangle.

c) Determine the coordinates of two points, C and D, that would make ABCD a square.

18. $\overline{AM}$ is a median. Show that $\overline{AM}$ is perpendicular to $\overline{BC}$.



19. ABCD is a rhombus. Show that the diagonals of the rhombus are perpendicular to each other.