4.5 Solving a Linear System Graphically

GOAL

Use a graph to solve a problem modelled by two linear relations.

YOU WILL NEED

- grid paper

Sari wants to join a website that allows its users to share music files. SHAREIT charges a $5 membership fee, plus $2.50 for each downloaded song. FILES 'R' US charges $3.00 per downloaded song.

? How can Sari determine which website she should join?

EXAMPLE 1 Solving a problem modelled by a system of linear equations

Determine the website that Sari should join.

Nick's Solution: Using a hand-drawn graph to solve a system of linear equations

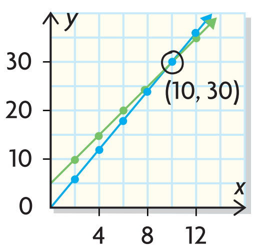
I chose x to represent the number of music files downloaded and y to represent the costs for each company.

SHAREIT: $y = 5 + 2.5x$

FILES 'R' US: $y = 3x$

I created a **system of linear equations** to model this situation. The total cost, y, in each case depends on the number of music files, x, that are downloaded. I used each equation to create a table of values.

Number of Songs, x	2	4	6	8	10	12
SHAREIT Cost, $y = 5 + 2.5x$	10	15	20	25	30	35
FILES'R'US Cost, $y = 3x$	6	12	18	24	30	36



I drew both linear relations on the same graph.

Since the coordinate (10, 30) lies on both lines, this is the **point of intersection** and the **solution to the system of equations**.

If Sari plans to download exactly 10 songs, it doesn't matter which site she purchases from. They will both charge $30.

system of linear equations
a set of equations (at least two) that represent linear relations between the same two variables

point of intersection
the point in common between two lines

solution to a system of linear equations
a point that satisfies both relations in a system of linear equations; the point of intersection represents an ordered pair that solves the system of linear equations

Check:

I checked that the point (10, 30) works in both relations.

SHAREIT	FILES 'R' US
$y = 5 + 2.5x$	$y = 3x$
$y = 5 + 2.5(10)$	$y = 3(10)$
$y = 5 + 25$	$y = 30$
$y = 30$	

Sari should choose FILES 'R' US if she plans to download fewer than 10 songs. If she is going to download more than 10 songs, she should choose SHAREIT.

From 1 to 9 songs downloaded, it costs less to purchase from FILES "R" US. I can tell because the graph is below the graph for SHAREIT. This switches when more than 10 songs are purchased.

Ben decided to use a graphing calculator to solve the problem.

Ben's Solution: Using technology to solve a system of linear equations

Tech Support

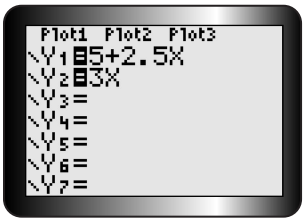
For help with using the graphing calculator to determine the point of intersection, see Appendix B-8.

I chose x to represent the number of music files downloaded and y to represent the costs for each company.

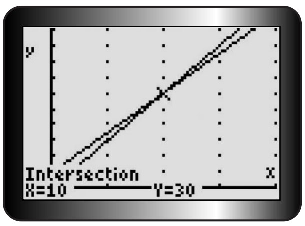
SHAREIT: $y = 5 + 2.5x$

FILES 'R' US: $y = 3x$

I created a **system of linear equations** to model this situation.



I used my calculator to graph both relations and determine the point of intersection.



The point of intersection is (10, 30).

Sari should choose FILES 'R' US if she plans to download fewer than 10 songs. If she is going to download more than 10 songs, she should choose SHAREIT.

Reflecting

A. Why is it reasonable for the point of intersection to be called the solution to a system of linear equations?

B. Which student's strategy would you prefer to use? Why?

APPLY the Math

EXAMPLE 2 Determining the point of intersection of two lines

What is the point of intersection of the graphs of

$3x + 2y = 12$ and $y = -\frac{3}{2}x + 2$?

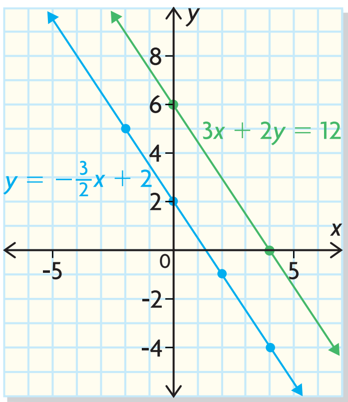
Julie's Solution

$3x + 2y = 12$	
x	y
4	0
0	6

$y = \frac{-3}{2}x + 2$	
x	y
-2	5
0	2
2	-1
4	-4

To graph $3x + 2y = 12$, I calculated the intercepts and connected the points to see the relation.

To graph $y = -\frac{3}{2}x + 2$, I used a table of values. I chose numbers for x that were divisible by 2 because the coefficient of x has 2 in its denominator. I substituted these into the equation to find the values for y. I connected the points to see the relation.
The lines are parallel, so they don't have a point in common.



There is no point of intersection.

EXAMPLE 3 Using a graphing strategy to estimate a break-even point

Jean just opened a new company that makes MP3 players.
He uses two equations to compare cost and revenue. In order for the company to break even, the cost must equal the revenue.

Cost equation:

- The company paid $5750 to set up the manufacturing line.
- The materials and labour cost for each machine is $50.
- The cost equation is $y = 50x + 5750$, where x is the number of MP3 players produced and y is the cost to produce x number of players.

Revenue equation:

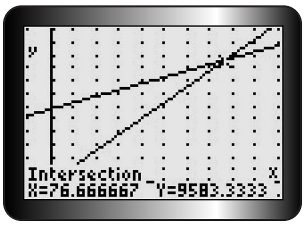
- The company sells each player for $125.
- The revenue equation is $y = 125x$, where x is the number of MP3 players sold and y is the revenue made for x number of MP3 players.

How many MP3 players must the company sell to break even?

Mason's Solution

Number of MP3 Players	Cost ($)	Revenue ($)
50	8 250	6 250
100	10 750	12 500
150	13 250	18 750

I created a table of values to find the break-even point. This is the point where the cost and the revenue are the same.
The cost is **more than** revenue when 50 MP3 players are sold.
The cost is **less than** the revenue when 100 MP3 players are sold.
This means that the break-even point must occur when more than 50 but fewer than 100 players are sold.



I used graphing technology to get a more accurate solution. I entered the equation $y = 50x + 5750$ in Y1 and $y = 125x$ in Y2. Then, I determined the point of intersection.

Intersection at approximately (76.67, 9583.33).

The break-even point is at (76.67, 9583.33).

Since you cannot sell part of an MP3 player, the company needs to sell about 77 MP3 players to break even.

If the company sells 77 MP3 players, its revenue and costs will be about $9600.

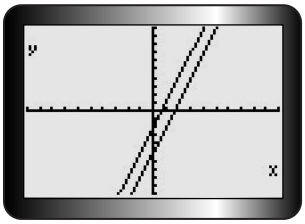
EXAMPLE 4 Determining the number of solutions to a system of linear equations

Determine the solution to each of the following:

a) $3x - y = 5$ and $y = 3x - 2$ **b)** $x - y = 2$ and $y = \dfrac{(2x - 4)}{2}$

Brent's Solution: Using graphing technology to solve the system of linear equations

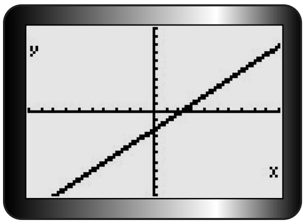
a)



There is no solution to this system of linear equations because the lines are parallel.

I used graphing technology to graph each linear relation. I could not find the point of intersection. I rewrote $3x - y = 5$ in its equivalent $y = mx + b$ form: $y = 3x - 5$. It appears that the slopes of the two lines are the same because the lines look parallel. This means that they do not intersect at any point.

b)



When I graphed both relations, the lines were identical.

$$x - y = 2$$
$$2 \times (x - y) = 2 \times 2$$
$$2x - 2y = 4$$

I realized that the equations were equivalent to each other. When I multiplied all the terms in the first equation by 2, I got the second equation.

There is an infinite number of solutions for this system of linear equations.

Since the equations are equivalent, all values that lie on one line also lie on the other line.

EXAMPLE 5 Using technology to solve a system of linear equations

Find two numbers where:

- The sum of both numbers divided by 4 is 3.
- Two times the difference of the two numbers is –36.

Donna's Solution

Let x represent one number and y represent the other.

$$\frac{(x + y)}{4} = 3 \qquad 2(x - y) = -36$$

I created a system of linear equations to model the two statements.

$$4 \times \frac{(x + y)}{4} = 3 \times 4 \qquad \frac{2(x - y)}{2} = \frac{-36}{2}$$

$$(x + y) = 12 \qquad x - y = -18$$

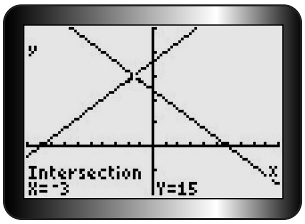
$$x + y - x = 12 - x \qquad x - y + y = -18 + y$$

$$y = 12 - x \qquad x = -18 + y$$

$$y = -x + 12 \qquad x + 18 = -18 + y + 18$$

$$x + 18 = y$$

I used inverse operations to rearrange the relations into the form $y = mx + b$, so that I could graph both relations on a graphing calculator.



I entered both relations into the equation editor and graphed the relations. I found the point of intersection using the graphing calculator. The point of intersection is $(-3, 15)$.

The numbers are -3 and 15.

Check:

Since the point of intersection satisfies both equations, the numbers are correct.

$$\frac{(x + y)}{4} = 3$$

Left Side	Right Side
$\frac{(-3 + 15)}{4}$	3
$= \frac{12}{4}$	
$= 3$	

$$2(x - y) = -36$$

Left Side	Right Side
$2(-3 - 15)$	-36
$= 2(-18)$	
$= -36$	

In Summary

Key Ideas

- When solving a system of linear relations, the point of intersection of their graphs is the solution to that system of linear equations.

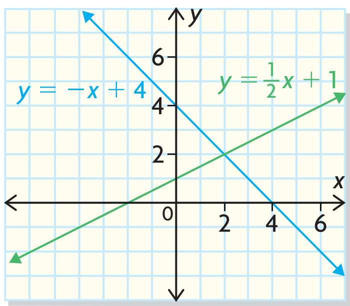
Need To Know

- The coordinates of the point of intersection can be estimated by graphing the relations by hand.
- Graphing technology helps determine the point of intersection with greater accuracy than is possible with a hand-drawn graph.
- A system of linear equations can have one point of intersection, zero points of intersection (if the graphs are parallel), or infinite points of intersection (if they are equivalent equations).

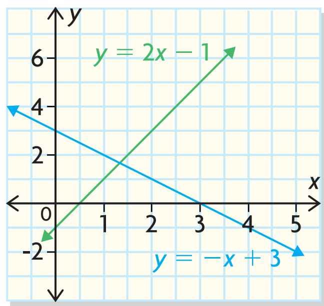
CHECK Your Understanding

1. Determine the point of intersection for each system of linear equations shown below.

a) $y = \frac{1}{2}x + 1$ and $y = -x + 4$



c) $y = 2x - 1$ and $y = -x + 3$

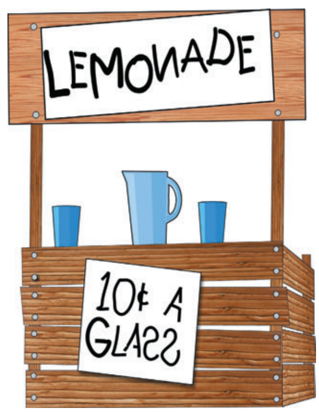


b) $y = x + 1$ and $y = 4x - 5$

d) $y = x$ and $y = -x$

2. Bill wants to earn extra money selling lemonade in front of his house. It costs $1.20 to start his business and each glass of lemonade costs $0.06 to make. He plans to sell the lemonade for $0.10 a glass.

a) Write an equation that represents his cost.
b) Write an equation that represents his revenue.
c) Graph both equations on the same set of axes.
d) What does the point of intersection mean in this case?
e) Does Bill make a profit or lose money for
  i) 20 glasses sold?
  ii) 35 glasses sold?
  iii) 50 glasses sold?



PRACTISING

3. Determine the point of intersection of each pair of lines.
   a) $y = -3x - 2$ and $2x + 3y = 5$
   b) $2x + 4y = 7$ and $-x + 0.75y = 5$
   c) $0.25x - 0.5y = 1$ and $3.25x + 4y = 22.5$
   d) $y = 3x + 6$ and $1 = 3x - y$

4. The sum of two integers is 42. The difference of the two numbers is 17.
   a) Create a system of linear equations to model each statement above.
   b) Determine the integers using a graph.

5. Mike has \$9.85 in dimes and quarters. If there are 58 coins altogether, how many dimes and how many quarters does Mike have?

6. Does each pair of lines intersect at the given point?
   a) $(2, 3)$: $y = x + 1$, $y = 4x - 5$
   b) $(1, -1)$: $y = 5x - 4$, $y = 2x - 3$
   c) $(0, 2)$: $y = 3x + 2$, $y = 5x - 1$
   d) $(-1, -3)$: $y = 4x + 1$, $y = x - 5$

7. Given the lines $y = 2$ and $y = 4x + 9$,
   a) Determine the point of intersection using a graph.
   b) Create the linear equation that you would solve to determine the x-value of the point of intersection.
   c) Solve the linear equation in part b) to verify your solution from part a).

8. Determine the point of intersection of each pair of lines:
   K
   a) $y - x = 9$ and $x - \frac{1}{6}y = -\frac{2}{3}$
   b) $y = 2$ and $y = 5$
   c) $2x - y = 0$ and $y = 5 + 2x$
   d) $y = -4$ and $x = 1$

9. Marie charges \$3 for every 4 bottles of water purchased from her store. She pays her supplier \$0.25 per bottle, plus \$250 for shelving and water delivery.
   a) Create a system of two linear equations to model this situation.
   b) How many bottles of water does she need to sell to break even?

10. **A** Mr. Smith is trying to decide which Internet service provider (ISP) to use for his home computer. UPLINK offers a flat fee of \$19 per month; BLUELINE offers a fee of \$10 per month, but charges \$0.59 per hour after the first 30 hours.
    a) Write the linear relation that models the cost in relation to the number of hours used for each plan.
    b) Estimate the point at which the costs for both companies would be the same.

c) What equation would you set up and solve to determine the exact point at which the costs would be the same? Why is this equation reasonable?

d) What advice would you give to Mr. Smith about which ISP to choose?

11. Mrs. Smith was trying to help her husband decide which ISP to use and she investigated two other companies on her own:
DOWNLINK offers a plan of \$5 per month plus \$1.15 per hour after the first 20 hours.
REDLINE offers a plan of \$2.50 per month plus \$1.80 per hour after the first 10 hours.
Should the Smiths consider either of these two companies in their decision? Why or why not?

12. Determine the point of intersection of each pair of lines:

a) $5x + 8y - 12 = 0$ and $-5x + 16y - 12 = 0$

b) $4x + y - 2 = 0$ and $8x + 2y - 4 = 0$

c) $\frac{1}{3}x - \frac{2}{5}y + \frac{1}{4} = 0$ and $2x - \frac{1}{7}y + \frac{1}{2} = 0$

d) $5x - 2.5y = 10$ and $3.1x + 4y = 6.2$

13. Given the relation $x + y = 5$, determine a second relation that:

T a) intersects $x + y = 5$ at $(2, 3)$

b) *does not* intersect $x + y = 5$

14. Movies to Go rents DVDs for \$2.50 and has no membership fee. Films
C 'R' Us rents videos for \$2 but has a \$10 membership fee. What advice would you give to someone who is deciding which video store to use?

15. Why does a system of two linear equations usually have only one solution for each of the two variables?

Extending

16. To determine the point of intersection of $y = 2x + 5$ and $y = 4x - 3$, Elena wrote $2x + 5 = 4x - 3$ and solved the equation. Why is this a reasonable strategy for determining the point of intersection of the two lines?

17. Compare the strategies of solving $3x + 4 = 5x + 3$ by using inverse operations and by graphing the two relations.

18. a) Determine the point(s) of intersection of $y = 2x^2$ and $y = 8$ using a graph.

b) Create and solve the equation that you would use to determine where the point of intersection lies.

c) Are your solutions from parts a) and b) the same? Explain.