R–8 Transformations of Functions

You can graph functions of the form $y = af(k(x - d)) + c$ by applying the appropriate transformations to key points on the parent function $y = f(x)$. Stretches/compressions and reflections (based on a and k) must be applied before translations (based on c and d).

The value of a determines whether there is a vertical stretch or compression and whether there is a reflection in the x-axis. The y-coordinate of each point is multiplied by a.

- If $|a| > 1$, the graph of $y = f(x)$ is stretched vertically by the factor $|a|$.
- If $0 < |a| < 1$, the graph is compressed vertically by the factor $|a|$.
- If a is negative, the graph is also reflected in the x-axis.

The value of k determines whether there is a horizontal stretch or compression and whether there is a reflection in the y-axis. The x-coordinate of each point is multiplied by $\frac{1}{k}$.

- If $|k| > 1$, the graph of $y = f(x)$ is compressed horizontally by the factor $\frac{1}{|k|}$.
- If $0 < |k| < 1$, the graph is stretched horizontally by the factor $\frac{1}{|k|}$.
- If k is negative, the graph is also reflected in the y-axis.

The value of c determines the vertical translation. This value is added to the y-coordinate of each point.

- If $c > 0$, the graph is translated c units up.
- If $c < 0$, the graph is translated c units down.

The value of d determines the horizontal translation. This value is added to the x-coordinate of each point.

- If $d > 0$, the graph is translated d units to the right.
- If $d < 0$, the graph is translated d units to the left.

EXAMPLE 1

What transformations to the parent function $y = f(x)$ would you perform to create the graph of $y = -2f(3(x - 4)) - 5$? What happens to the coordinates of each point on the parent function?

Solution

Comparing the transformed function with the general form $y = af(k(x - d)) + c$, we have $a = -2$, $k = 3$, $d = 4$, and $c = -5$.

EXAMPLE 2

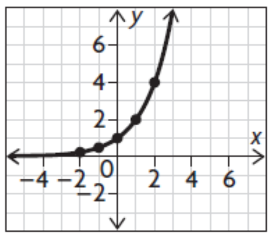
Graph the function $y = \frac{3}{2} \times 2^{-\frac{1}{2}(x-3)} + 1$ by applying of the appropriate transformations to the parent function $y = 2x$.

Solution

Table of values for $y = 2^x$

x	y
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4

Graph of $y = 2^x$



We start with points on the parent function $y = 2^x$.

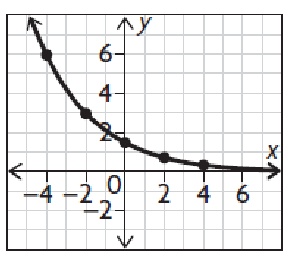
the x-axis. The y-coordinate of each point is multiplied by -2.

- Since $k = 3$, there is a horizontal compression by a factor of $\frac{1}{3}$. The x-coordinate of each point is multiplied by $\frac{1}{3}$.
- Since $c = -5$, there is a vertical translation 5 units down. The value -5 is added to the y-coordinate of each point.
- Since $d = 4$, there is a horizontal translation 4 units to the right. The value 4 is added to the x-coordinate of each point.

Table of values for $y = \frac{3}{2} \times 2^{-\frac{1}{2}x}$

x	y
4	$\frac{3}{8}$
2	$\frac{3}{4}$
0	$\frac{3}{2}$
-2	3
-4	6

Graph of $y = \frac{3}{2} \times 2^{-\frac{1}{2}x}$



Apply any stretches/compressions and reflections next.

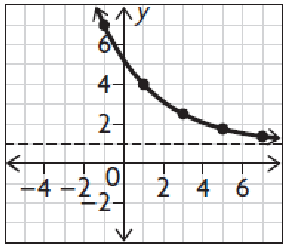
Since $a = \frac{3}{2}$, there is a vertical stretch. Each y-coordinate is multiplied by $\frac{3}{2}$.

Since $k = -\frac{1}{2}$ there is a horizontal stretch and also a reflection in the y-axis. Each x-coordinate is multiplied by -2.

Table of values for $y = \frac{3}{2} \times 2^{-\frac{1}{2}(x-3)} + 1$

x	y
7	$1\frac{3}{8}$
5	$1\frac{3}{4}$
3	$2\frac{1}{2}$
1	4
−1	7

Graph of $y = \frac{3}{2} \times 2^{-\frac{1}{2}(x-3)} + 1$



Apply any translations last.
Since $c = 1$, there is a translation up. The value 1 is added to each y-coordinate.
Since $d = 3$, there is a translation to the right. The value 3 is added to each x-coordinate.

Notice that the horizontal asymptote is shifted up to $y = 1$.

Practising

1. Describe the transformations that you would apply to the graph of $y = f(x)$ to graph each of the following functions.

a) $y = 3f(x) - 2$

b) $y = f\left(\frac{1}{2}(x + 3)\right)$

c) $y = f(2x) + 7$

d) $y = -3f(2(x - 1)) - 2$

e) $y = -f(-x) + 4$

f) $y = -\frac{1}{5}f(-x) - 3$

2. The point (2, 5) is on the graph of $y = f(x)$. State the coordinates of the image of this point under each of the following transformations.

a) $y = f(3x)$

b) $y = -2f(x)$

c) $y = f(x - 4)$

d) $y = f(x) + 7$

3. Given the function $f(x) = x^2$, state the equation of the transformed function under a vertical stretch of factor 3, a reflection in the x-axis, a horizontal translation 3 units to the right, and a vertical translation 2 units up.

4. Consider the function $f(x) = x^3$.

a) Make a table of values for f using $x \in \{-2, -1, 0, 1, 2\}$.

b) Describe the transformations to f that result in the function $g(x) = \frac{1}{2}(x - 4)^3 + 5$.

c) Determine the five points on the graph of g that are the images of the five points in your table of values for f in part a).

5. Consider the functions $Y_1 = \sqrt{x}$ and $Y_2 = \sqrt{4 - x}$. What transformations to Y_1 result in Y_2?