A–1 Operations with Integers

Set of integers $\mathbf{I} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$

Addition

To add two integers,

- if the signs are the same, then the sum has the same sign as well:
  $(-12) + (-5) = -17$
- if the signs are different, then the sum takes the sign of the larger number:
  $18 + (-5) = 13$

Subtraction

Add the opposite:

$$\begin{aligned} -15 - (-8) &= -15 + 8 \\ &= -7 \end{aligned}$$

Multiplication and Division

To multiply or divide two integers,

- if the two integers have the same sign, then the answer is positive:
  $6 \times 8 = 48$, $(-36) \div (-9) = 4$
- if the two integers have different signs, then the answer is negative:
  $(-5) \times 9 = -45$, $54 \div (-6) = -9$

More Than One Operation

Follow the order of operations.

B	Brackets	
E	Exponents	
D	Division	from left to right
M	Multiplication	
A	Addition	from left to right
S	Subtraction	

EXAMPLE

Evaluate.

a) $-10 + (-12)$

b) $(-12) + 7$

c) $(-11) + (-4) + 12 + (-7) + 18$

d) $(-6) \times 9 \div 3$

e) $\dfrac{20 + (-12) \div (-3)}{(-4 + 12) \div (-2)}$

Solution

a) $-10 + (-12) = -22$
b) $(-12) + 7 = -5$
c) $(-11) + (-4) + 12 + (-7) + 18$
$= (-22) + 30$
$= 8$
d) $(-6) \times 9 \div 3$
$= -54 \div 3$
$= -18$

e) $\dfrac{20 + (-12) \div (-3)}{(-4 + 12) \div (-2)}$

$= \dfrac{20 + 4}{8 \div (-2)}$

$= \dfrac{24}{-4}$

$= -6$

Practising

1. Evaluate.
a) $6 + (-3)$
b) $12 - (-13)$
c) $-17 - 7$
d) $(-23) + 9 - (-4)$
e) $24 - 36 - (-6)$
f) $32 + (-10) + (-12) - 18 - (-14)$

2. Which choice would make each statement true: $>$, $<$, or $=$?
a) $-5 - 4 - 3 + 3$ ■ $-4 - 3 - 1 - (-2)$
b) $4 - 6 + 6 - 8$ ■ $-3 - 5 - (-7) - 4$
c) $8 - 6 - (-4) - 5$ ■ $5 - 13 - 7 - (-8)$
d) $5 - 13 + 7 - 2$ ■ $4 - 5 - (-3) - 5$

3. Evaluate.
a) $(-11) \times (-5)$
b) $(-3)(5)(-4)$
c) $35 \div (-5)$
d) $(-72) \div (-9)$
e) $(5)(-9) \div (-3)(7)$
f) $56 \div [(8)(7)] \div 49$

4. Evaluate.
a) $(-3)^2 - (-2)^2$
b) $(-5)^2 - (-7) + (-12)$
c) $-4 + 20 \div (-4)$
d) $-3(-4) + 8^2$
e) $(-16) - [(-8) \div 2]$
f) $8 \div (-4) + 4 \div (-2)^2$

5. Evaluate.

a) $\dfrac{-12 - 3}{-3 - 2}$

b) $\dfrac{-18 + 6}{(-3)(-4)}$

c) $\dfrac{(-16 + 4) \div 2}{8 \div (-8) + 4}$

d) $\dfrac{-5 + (-3)(-6)}{(-2)^2 + (-3)^2}$

A–2 Operations with Rational Numbers

Set of rational numbers $\mathbf{Q} = \left\{ \frac{a}{b} \middle| a, b \in \mathbf{I}, b \neq 0 \right\}$

Addition and Subtraction
To add or subtract rational numbers, you need to find a common denominator.

Multiplication
$\frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd}$, but first reduce to lowest terms where possible.

Division
To divide by a rational number, multiply by the reciprocal.

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$$

More Than One Operation
Follow the order of operations.

EXAMPLE 1

Simplify $\frac{-2}{5} + \frac{3}{-2} - \frac{3}{10}$.

Solution

$$\frac{-2}{5} + \frac{3}{-2} - \frac{3}{10} = \frac{-4}{10} + \frac{-15}{10} - \frac{3}{10}$$
$$= \frac{-4 - 15 - 3}{10}$$
$$= \frac{-22}{10}$$
$$= -\frac{11}{5} \text{ or } -2\frac{1}{5}$$

EXAMPLE 2

Simplify $\frac{3}{4} \times \frac{-4}{5} \div \frac{-3}{7}$.

Solution

$$\frac{3}{4} \times \frac{-4}{5} \div \frac{-3}{7} = \frac{3}{4} \times \frac{-4}{5} \times \frac{7}{-3}$$
$$= \frac{\cancel{3}^{1}}{\cancel{4}_{1}} \times \frac{-\cancel{4}^{-1}}{5} \times \frac{7}{-\cancel{3}_{-1}}$$
$$= \frac{7}{5} \text{ or } 1\frac{2}{5}$$

Practising

1. Evaluate.

a) $\frac{1}{4} + \frac{-3}{4}$

b) $\frac{1}{2} - \frac{-2}{3}$

c) $\frac{-1}{4} - 1\frac{1}{3}$

d) $-8\frac{1}{4} - \frac{-1}{-3}$

e) $\frac{-3}{5} + \frac{-3}{4} - \frac{7}{10}$

f) $\frac{2}{3} - \frac{-1}{2} - \frac{1}{6}$

2. Evaluate.

a) $\frac{4}{5} \times \frac{-20}{25}$

b) $\frac{3}{-2} \times \frac{6}{5}$

c) $\left(\frac{-1}{3}\right)\left(\frac{2}{-5}\right)$

d) $\left(\frac{9}{4}\right)\left(\frac{-2}{-3}\right)$

e) $\left(-1\frac{1}{10}\right)\left(3\frac{1}{11}\right)$

f) $-4\frac{1}{6} \times \left(-7\frac{3}{4}\right)$

3. Evaluate.

a) $\frac{-4}{3} \div \frac{2}{-3}$

b) $-7\frac{1}{8} \div \frac{3}{2}$

c) $\frac{-2}{3} \div \frac{-3}{8}$

d) $\frac{-3}{-2} \div \left(\frac{-1}{3}\right)$

e) $-6 \div \left(\frac{-4}{5}\right)$

f) $\left(-2\frac{1}{3}\right) \div \left(-3\frac{1}{2}\right)$

4. Simplify.

a) $\frac{-2}{5} - \left(\frac{-1}{10} + \frac{1}{-2}\right)$

b) $\frac{-3}{5}\left(\frac{-3}{4} - \frac{-1}{4}\right)$

c) $\left(\frac{3}{5}\right)\left(\frac{1}{-6}\right)\left(\frac{-2}{3}\right)$

d) $\left(\frac{-2}{3}\right)^2\left(\frac{1}{-2}\right)^3$

e) $\left(\frac{-2}{5} + \frac{1}{-2}\right) \div \left(\frac{5}{-8} - \frac{-1}{2}\right)$

f) $\dfrac{\frac{-4}{5} - \frac{-3}{5}}{\frac{1}{3} - \frac{-1}{5}}$

A–3 Exponent Laws

exponent

3^4 and a^n are called powers

base

4 factors of 3 $\quad$ n factors of a

$3^4 = (3)(3)(3)(3)$ $\quad$ $a^n = (a)(a)(a)\ldots(a)$

Operations with powers follow a set of procedures or rules.

Rule	Description	Algebraic Description	Example
Zero as an Exponent	When an exponent is zero, the value of the power is 1.	$a^0 = 1$	$120^0 = 1$
Negative Exponents	A negative exponent is the reciprocal of the power with a positive exponent.	$a^{-n} = \frac{1}{a^n}$	$3^{-2} = \frac{1}{3^2}$ $= \frac{1}{9}$
Multiplication	When the bases are the same, keep the base the same and add exponents.	$(a^m)(a^n) = a^{m+n}$	$(5^4)(5^{-3}) = 5^{4+(-3)}$ $= 5^{4-3}$ $= 5^1$ $= 5$
Division	When the bases are the same, keep the base the same and subtract exponents.	$\frac{a^m}{a^n} = a^{m-n}$	$\frac{4^6}{4^{-2}} = 4^{6-(-2)}$ $= 4^{6+2}$ $= 4^8$
Power of a Power	Keep the base and multiply the exponents.	$(a^m)^n = a^{mn}$	$(3^2)^4 = 3^{(2)(4)}$ $= 3^8$

EXAMPLE

Simplify and evaluate.

$3(3^7) \div (3^3)^2$

Solution

$$\begin{aligned}
3(3^7) \div (3^3)^2 &= 3^{1+7} \div 3^{3\times 2} \\
&= 3^8 \div 3^6 \\
&= 3^{8-6} \\
&= 3^2 \\
&= 9
\end{aligned}$$

Practising

1. Evaluate to three decimal places where necessary.

a) 4^2

b) 5^0

c) 3^2

d) -3^2

e) $(-5)^3$

f) $\left(\frac{1}{2}\right)^3$

2. Evaluate.

a) $3^0 + 5^0$

b) $2^2 + 3^3$

c) $5^2 - 4^2$

d) $\left(\frac{1}{2}\right)^3\left(\frac{2}{3}\right)^2$

e) $-2^5 + 2^4$

f) $\left(\frac{1}{2}\right)^2 + \left(\frac{1}{3}\right)^2$

3. Evaluate to an exact answer.

a) $\frac{9^8}{9^7}$

b) $\frac{2(5^5)}{5^3}$

c) $(4^5)(4^2)^3$

d) $\frac{(3^2)(3^3)}{(3^4)^2}$

4. Simplify.

a) $(x)^5(x)^3$

b) $(m)^2(m)^4(m)^3$

c) $(y)^5(y)^2$

d) $(a^b)^c$

e) $\frac{(x^5)(x^3)}{x^2}$

f) $\left(\frac{x^4}{y^3}\right)^3$

5. Simplify.

a) $(x^2y^4)(x^3y^2)$

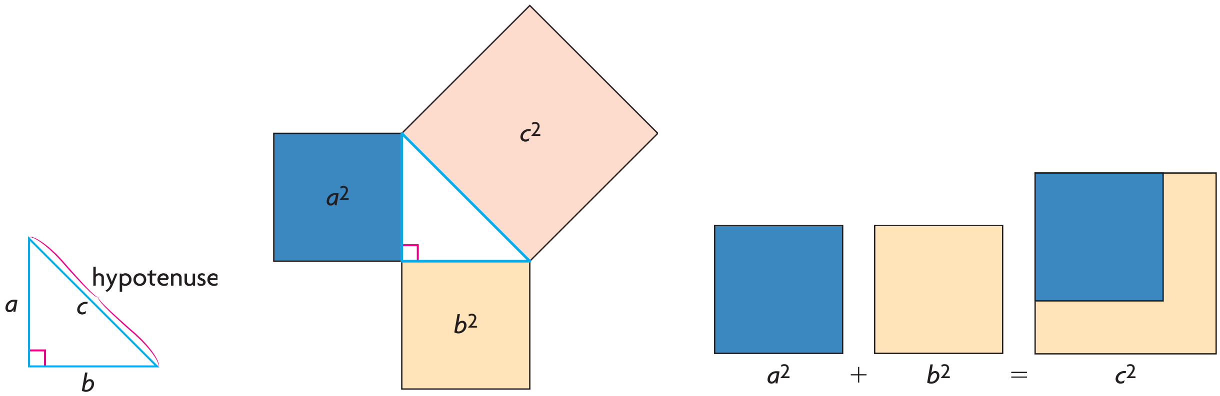
b) $(-2m^3)^2(3m^2)^3$

c) $\frac{(5x^2)^2}{(5x^2)^0}$

d) $(4u^3v^2)^2 \div (-2u^2v^3)^2$

A–4 The Pythagorean Theorem

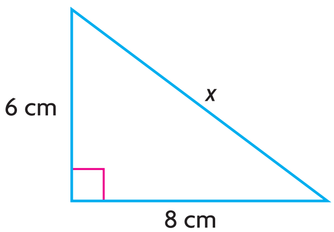
The three sides of a right triangle are related to each other in a unique way. Every right triangle has a longest side, called the **hypotenuse**, which is always opposite the right angle. One of the important relationships in mathematics is known as the **Pythagorean theorem**. It states that the area of the square of the hypotenuse is equal to the sum of the areas of the squares of the other two sides.



Practising

1. For each right triangle, write the equation for the Pythagorean theorem.

a)



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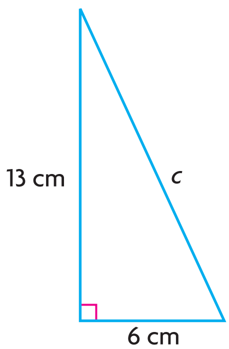

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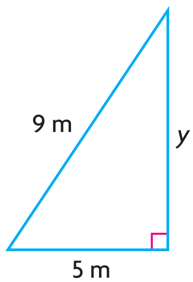

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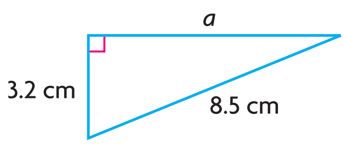
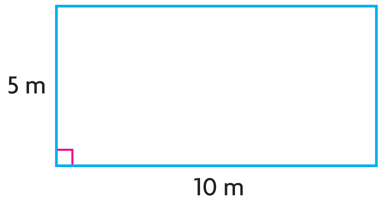


2. Calculate the length of the unknown side of each triangle in question 1. Round all answers to one decimal place.

3. Find the value of each unknown measure to the nearest hundredth.
   a) $a^2 = 5^2 + 13^2$
   b) $10^2 = 8^2 + m^2$
   c) $26^2 = b^2 + 12^2$
   d) $2.3^2 + 4.7^2 = c^2$

4. Determine the length of the diagonals of each rectangle to the nearest tenth.

a)



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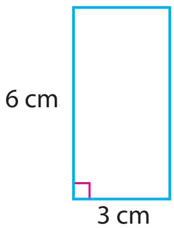
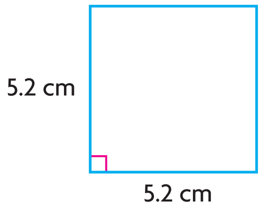
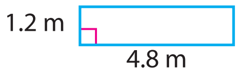


c)



d)



5. An isosceles triangle has a hypotenuse 15 cm long. Determine the length of the two equal sides.

6. An apartment building casts a shadow. From the tip of the shadow to the top of the building is 100 m. The tip of the shadow is 72 m from the base of the building. How tall is the building?

A–5 Graphing Linear Relationships

The graph of a linear relationship ($Ax + By + C = 0$) is a straight line. The graph can be drawn if at least two ordered pairs of the relationship are known. This information can be determined in several different ways.

EXAMPLE 1 Table of Values

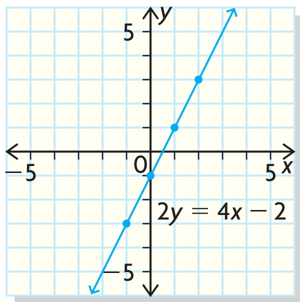
Sketch the graph of $2y = 4x - 2$.

Solution

A table of values can be created. Express the equation in the form $y = mx + b$.

$$\frac{2y}{2} = \frac{4x - 2}{2}$$
$$y = 2x - 1$$

x	y
-1	$2(-1) - 1 = -3$
0	$2(0) - 1 = -1$
1	$2(1) - 1 = 1$
2	$2(2) - 1 = 3$



EXAMPLE 2 Using Intercepts

Sketch the graph of $2x + 4y = 8$.

Solution

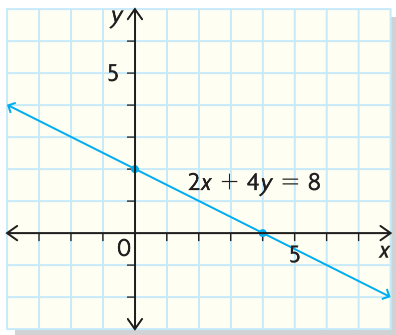
The intercepts of the line can be found.
For the x-intercept, let $y = 0$.

$$2x + 4(0) = 8$$
$$2x = 8$$
$$x = 4$$

x	y
4	0
0	2

For the y-intercept, let $x = 0$.

$$2(0) + 4y = 8$$
$$4y = 8$$
$$y = 2$$

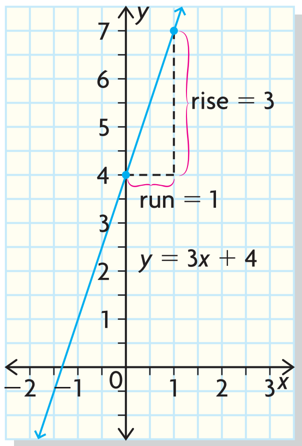


EXAMPLE 3 Using the Slope and y-intercept

Sketch the graph of $y = 3x + 4$.

Solution

When the equation is in the form $y = mx + b$, the slope, m, and y-intercept, b, can be determined. For $y = 3x + 4$, the line has a slope of 3 and a y-intercept of 4.



Practising

1. Express each equation in the form $y = mx + b$.
   a) $3y = 6x + 9$
   b) $2x - 4y = 8$
   c) $3x + 6y - 12 = 0$
   d) $5x = y - 9$

2. Graph each equation, using a table of values where $x \in \{-2, -1, 0, 1, 2\}$.
   a) $y = 3x - 1$
   b) $y = \frac{1}{2}x + 4$
   c) $2x + 3y = 6$
   d) $y = 4$

3. Determine the x- and y-intercepts of each equation.
   a) $x + y = 10$
   b) $2x + 4y = 16$
   c) $50 - 10x - y = 0$
   d) $\frac{x}{2} + \frac{y}{4} = 1$

4. Graph each equation by determining the intercepts.
   a) $x + y = 4$
   b) $x - y = 3$
   c) $2x + 5y = 10$
   d) $3x - 4y = 12$

5. Graph each equation, using the slope and y-intercept.
   a) $y = 2x + 3$
   b) $y = \frac{2}{3}x + 1$
   c) $y = -\frac{3}{4}x - 2$
   d) $2y = x + 6$

6. Graph each equation. Use the most suitable method.
   a) $y = 5x + 2$
   b) $3x - y = 6$
   c) $y = -\frac{2}{3}x + 4$
   d) $4x = 20 - 5y$

A–6 Solving Linear Systems

Many kinds of situations can be modelled with linear equations. When two or more linear equations are used to model a problem, they are called a linear system of equations. Point $P(x, y)$ is the intersection point of the linear equations in the system. Point P is called the solution of the linear system and satisfies all equations in the system.

Solving a Linear System Graphically

Linear systems can be solved graphically, although this method does not always yield an exact solution.

EXAMPLE 1

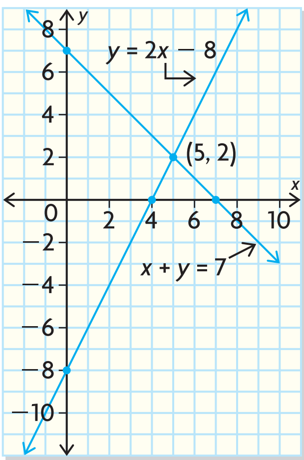
Solve the system graphically.

$y = 2x + 8$ ①

$x + y = 7$ ②

Solution

Draw both graphs on the same axes and locate the point of intersection.
Point (5, 2) appears to be the point of intersection. Verify this result algebraically by substituting (5, 2) into equations ① and ②.



In equation ①,

L.S.	R.S.
$y = 2$	$2x - 8$
	$= 2(5) - 8$
	$= 2$

Therefore, L.S. = R.S.

In equation ②,

L.S.	R.S.
$x + y$	7
$= 5 + 2$	
$= 7$	

Therefore, L.S. = R.S.

Solving a System of Linear Equations by Substitution

Linear systems can also be solved by using algebra. Algebraic methods always yield exact solutions. One such method is called substitution.

EXAMPLE 2

Solve the system of linear equations by substitution.

$3x + 2y + 24 = 0$ ①

$5y + 2x = -38$ ②

Solution

Solve for y in equation ①. $$3x + 2y + 24 = 0 \quad ①$$ $$2y = -24 - 3x$$ $$y = -12 - \frac{3}{2}x$$	Choose one of the equations and isolate one of its variables by expressing that variable in terms of the other variable.
Substitute $y = -12 - \frac{3}{2}x$ into equation ②. $$5y + 2x = -38 \quad ②$$ $$5\left(-12 - \frac{3}{2}x\right) + 2x = -38$$ $$-60 - \frac{15}{2}x + 2x = -38$$ $$\frac{-15x + 4x}{2} = -38 + 60$$ $$\frac{-11x}{2} = 22$$ $$-11x = 44$$ $$x = -4$$	Substitute the expression that you determined for the corresponding variable in the other equation.
Substitute $x = -4$ into equation ①. $$3x + 2y + 24 = 0 \quad ①$$ $$3(-4) + 2y + 24 = 0$$ $$-12 + 2y + 24 = 0$$ $$2y + 12 = 0$$ $$2y = -12$$ $$y = -6$$	Determine the other value by substituting the solved value into equation ① or ②.

The solution of the system is $(-4, -6)$.

Practising

1. Determine which ordered pair satisfies both equations.

a) $x + y = 5$ $\quad$ $(4, 1), (2, 3),$
$x = y + 1$ $\quad$ $(3, 2), (5, 4)$

b) $x + y = -5$ $\quad$ $(3, -6), (10, -5),$
$y = -2x$ $\quad$ $(5, -10), (-3, -2)$

2. Solve the system by drawing the graph.

a) $3x + 4y = 12$
$2x + 3y = 9$

b) $x + y = -4$
$2x - y = 4$

c) $x - 3y + 1 = 0$
$2x + y - 4 = 0$

d) $x = 1 - 2y$
$y = 2x + 3$

3. Using substitution to determine the coordinates of the point of intersection.

a) $3p + 2q - 1 = 0$
$p = q + 2$

b) $2m - n = 3$
$m + 2n = 24$

c) $2x + 5y + 18 = 0$
$x + 2y + 6 = 0$

d) $6g - 3h = 9$
$4g = 5 + 3h$

e) $10x + 15y = 30$
$15x - 5y = -25$

f) $13a - 7b = -11$
$a + 5b = 13$

A–7 Evaluating Algebraic Expressions and Formulas

Algebraic expressions and formulas are evaluated by substituting the given numbers for the variables. Then follow the order of operations to calculate the answer.

EXAMPLE 1

Find the value of $2x^2 - y$ if $x = -2$ and $y = 3$.

Solution

$$\begin{aligned} 2x^2 - y &= 2(-2)^2 - 3 \\ &= 2(4) - 3 \\ &= 8 - 3 \\ &= 5 \end{aligned}$$

EXAMPLE 2

The formula for finding the volume of a cylinder is $V = \pi r^2 h$. Find the volume of a cylinder with a radius of 2.5 cm and a height of 7.5 cm.

Solution

$$\begin{aligned} V &= \pi r^2 h \\ &\doteq (3.14)(2.5)^2(7.5) \\ &= (3.14)(6.25)(7.5) \\ &\doteq 147 \text{ cm}^3 \end{aligned}$$

Practising

1. Find the value of each expression for $x = -5$ and $y = -4$.

a) $-4x - 2y$
b) $-3x - 2y^2$
c) $(3x - 4y)^2$
d) $\left(\frac{x}{y}\right) - \left(\frac{y}{x}\right)$

2. If $x = -\frac{1}{2}$ and $y = \frac{2}{3}$, find the value of each expression.

a) $x + y$
b) $x + 2y$
c) $3x - 2y$
d) $\frac{1}{2}x - \frac{1}{2}y$

3. **a)** The formula for the area of a triangle is $A = \frac{1}{2}bh$. Find the area of a triangle when $b = 13.5$ cm and $h = 12.2$ cm.

b) The area of a circle is found using the formula $A = \pi r^2$. Find the area of a circle with a radius of 4.3 m.

c) The hypotenuse of a right triangle, c, is found using the formula $c = \sqrt{a^2 + b^2}$. Find the length of the hypotenuse when $a = 6$ m and $b = 8$ m.

d) A sphere's volume is calculated using the formula $V = \frac{4}{3}\pi r^3$. Determine the volume of a sphere with a radius of 10.5 cm.

A–8 Expanding and Simplifying Algebraic Expressions

Type	Description	Example
Collecting Like Terms $2a + 3a = 5a$	Add or subtract the coefficients of the terms that have the same variables and exponents.	$3a - 2b - 5a + b$ $= 3a - 5a - 2b + b$ $= -2a - b$
Distributive Property $a(b + c) = ab + ac$	Multiply each term of the binomial by the monomial.	$-4a(2a - 3b)$ $= -8a^2 + 12ab$
Product of Two Binomials $(a + b)(c + d)$ $= ac + ad + bc + bd$	Multiply the first term of the first binomial by the second binomial, and then multiply the second term of the first binomial by the second binomial. Collect like terms if possible.	$(2x^2 - 3)(5x^2 + 2)$ $= 10x^4 + 4x^2 - 15x^2 - 6$ $= 10x^4 - 11x^2 - 6$

Practising

1. Simplify.
a) $3x + 2y - 5x - 7y$
b) $5x^2 - 4x^3 + 6x^2$
c) $(4x - 5y) - (6x + 3y) - (7x + 2y)$
d) $m^2n + p - (2p - 3m^2n)$

2. Expand.
a) $3(2x + 5y - 2)$
b) $5x(x^2 - x + y)$
c) $m^2(3m^2 - 2n)$
d) $x^5y^3(4x^2y^4 - 2xy^5)$

3. Expand and simplify.
a) $3x(x + 2) + 5x(x - 2)$
b) $-7h(2h + 5) - 4h(5h - 3)$
c) $2m^2n(m^3 - n) - 5m^2n(3m^3 + 4n)$
d) $-3xy^3(5x + 2y + 1) + 2xy^3(-3y - 2 + 7x)$

4. Expand and simplify.
a) $(3x - 2)(4x + 5)$
b) $(7 - 3y)(2 + 4y)$
c) $(5x - 7y)(4x + y)$
d) $(3x^3 - 4y^2)(5x^3 + 2y^2)$

A–9 Factoring Algebraic Expressions

Factoring is the opposite of expanding.

expanding ⟶

$2x(3x - 5) = 6x^2 - 10x$

⟵ factoring

Type	Example	Comment
Common Factoring $ab + ac = a(b + c)$ Factor out the largest common factor of each term.	$10x^4 - 8x^3 + 6x^5$ $= 2x^3(5x - 4 + 3x^2)$	Each term has a common factor of $2x^3$.
Factoring Trinomials $ax^2 + bx + c$, when $a = 1$ Write as the product of two binomials. Determine two numbers whose sum is b and whose product is c.	$x^2 + 4x - 21$ $= (x + 7)(x - 3)$	$(-21) = 7(-3)$ and $4 = 7 + (-3)$
Factoring Trinomials $ax^2 + bx + c$, when $a \neq 1$ Look for a common factor. If none exists, use decomposition and write as the product of two binomials. Check by expanding and simplifying.	$3x^2 + 4x - 4$ $= 3x^2 - 2x + 6x - 4$ $= (3x^2 - 2x) + (6x - 4)$ $= x(3x - 2) + 2(3x - 2)$ $= (3x - 2)(x + 2)$ Check: $(3x)(x) + (3x)(2)$ $+ (-2)(x) + (-2)(2)$ $= 3x^2 + 6x - 2x - 4$ $= 3x^2 + 4x - 4$	Multiply: $3(-4) = -12$ Find two numbers whose product is -12 and whose sum is 4. In this case, the numbers are 6 and -2. Using these numbers, decompose the x-term. Group the terms and factor out the common factors.

Practising

1. Factor each expression.

a) $4 - 8x$

b) $6x^2 - 5x$

c) $3m^2n^3 - 9m^3n^4$

d) $28x^2 - 14xy$

2. Factor each expression.

a) $x^2 - x - 6$

b) $x^2 + 7x + 10$

c) $x^2 - 9x + 20$

d) $3y^2 + 18y + 24$

3. Factor.

a) $6y^2 - y - 2$

b) $12x^2 + x - 1$

c) $5a^2 + 7a - 6$

d) $12x^2 - 18x - 12$

A–10 Solving Quadratic Equations Algebraically

To solve a quadratic equation, first rewrite it in the form $ax^2 + bx + c = 0$. Then factor the left side, if possible, or use the quadratic formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

A quadratic equation can have no roots, one root, or two roots. Not all quadratic equations can be solved by factoring.

EXAMPLE 1

Solve $x^2 + 3x = 10$.

Solution

$$x^2 + 3x = 10$$

$$x^2 + 3x - 10 = 0$$

$$(x + 5)(x - 2) = 0$$

Then $x + 5 = 0$ or $x - 2 = 0$

$x = -5$ or $x = 2$

EXAMPLE 2

Solve $-x = 3 - 2x^2$.

Solution

$$-x = 3 - 2x^2$$

$$2x^2 - x - 3 = 0$$

$a = 2$, $b = -1$, and $c = -3$

$$x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-3)}}{2(2)}$$

$$= \frac{1 \pm \sqrt{1 + 24}}{4}$$

$$= \frac{1 \pm \sqrt{25}}{4}$$

$$= \frac{1 \pm 5}{4}$$

$x = \frac{1 + 5}{4}$ or $x = \frac{1 - 5}{4}$

$x = \frac{6}{4} = \frac{3}{2}$ or $x = \frac{-4}{4} = -1$

Practising

1. Solve.
   a) $(x - 3)(x - 2) = 0$
   b) $(2x - 5)(3x - 1) = 0$
   c) $(m - 4)(m - 3) = 0$
   d) $(3 - 2x)(4 - 3x) = 0$
   e) $(2y + 5)(3y - 7) = 0$
   f) $(5n - 3)(4 - 3n) = 0$

2. Determine the roots.
   a) $x^2 - x - 2 = 0$
   b) $x^2 + x - 20 = 0$
   c) $m^2 + 2m - 15 = 0$
   d) $6x^2 - x - 2 = 0$
   e) $6t^2 + 5t - 4 = 0$
   f) $2x^2 + 4x - 30 = 0$

3. Solve.
   a) $4x^2 = 8x - 1$
   b) $4x^2 = 9$
   c) $6x^2 - x = 1$
   d) $5x^2 - 6 = -7x$
   e) $3x^2 + 5x - 1 = 2x^2 + 6x + 5$
   f) $7x^2 + 2(2x + 3) = 2(3x^2 - 4) + 13x$

4. A model rocket is shot straight into the air. Its height in metres at t seconds is given by $h = -4.9t^2 + 29.4t$. When does the rocket reach the ground?

5. The population of a city is modelled by $P = 0.5t^2 + 10t + 200$, where P is the population in thousands and t is the time in years, with $t = 0$ corresponding to the year 2000. When is the population 350 000?

A–11 Creating Scatter Plots and Lines or Curves of Good Fit

A **scatter plot** is a graph that shows the relationship between two sets of numeric data. The points in a scatter plot often show a general pattern, or **trend**. A line that approximates a trend for the data in a scatter plot is called a **line of best fit**. A line of best fit passes through as many points as possible, with the remaining points grouped equally above and below the line.

Data that have a **positive correlation** have a pattern that slopes up and to the right. Data that have a **negative correlation** have a pattern that slopes down and to the right. If the points nearly form a line, then the correlation is strong. If the points are dispersed, but still form some linear pattern, then the correlation is weak.

EXAMPLE 1

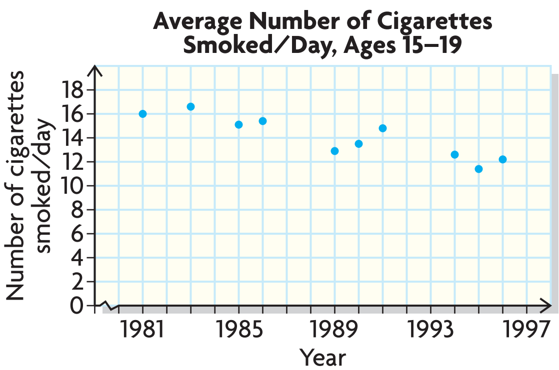
a) Make a scatter plot of the data and describe the kind of correlation the scatter plot shows.
b) Draw the line of best fit.

Long-Term Trends in Average Number of Cigarettes Smoked per Day by Smokers Aged 15–19

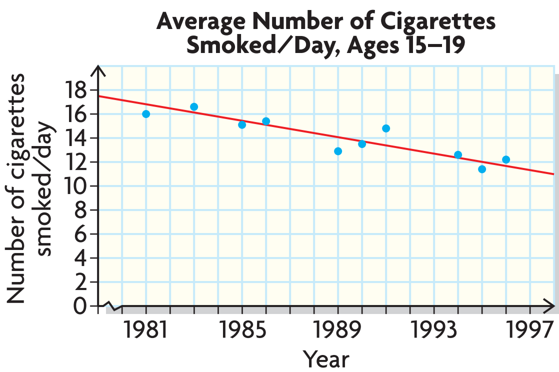
Year	1981	1983	1985	1986	1989	1990	1991	1994	1995	1996
Number Per Day	16.0	16.6	15.1	15.4	12.9	13.5	14.8	12.6	11.4	12.2

Solution

a)



b)



The scatter plot shows a negative correlation.

EXAMPLE 2

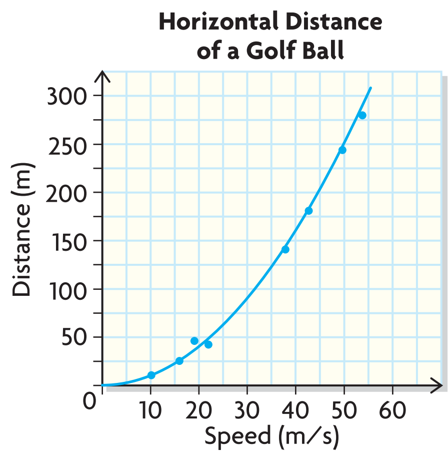
A professional golfer is taking part in a scientific investigation. Each time she drives the ball from the tee, a motion sensor records the initial speed of the ball. The final horizontal distance of the ball from the tee is also recorded. Here are the results:

Speed (m/s)	10	16	19	22	38	43	50	54
Distance (m)	10	25	47	43	142	182	244	280

Draw the line or curve of good fit.

Solution

The scatter plot shows that a line of best fit does not fit the data as well as an upward-sloping curve does. Therefore, sketch a curve of good fit.



Practising

1. For each set of data,

i) create a scatter plot and draw the line of best fit

ii) describe the type of correlation the trend in the data displays

a) Population of the Hamilton–Wentworth, Ontario, Region

Year	1966	1976	1986	1996	1998
Population	449 116	529 371	557 029	624 360	618 658

b) Percent of Canadians with Less than Grade 9 Education

Year	1976	1981	1986	1991	1996
Percent of the Population	25.4	20.7	17.7	14.3	12.4

2. In an experiment for a physics project, marbles are rolled up a ramp. A motion sensor detects the speed of the marble at the start of the ramp, and the final height of the marble is recorded. However, the motion sensor may not be measuring accurately. Here are the data:

Speed (m/s)	1.2	2.1	2.8	3.3	4.0	4.5	5.1	5.6
Final Height (m)	0.07	0.21	0.38	0.49	0.86	1.02	1.36	1.51

a) Draw a curve of good fit for the data.

b) How consistent are the motion sensor's measurements? Explain.

A–12 Using Properties of Quadratic Relations to Sketch Their Graphs

If the algebraic expression of a relation can be identified as quadratic, its graph can be sketched without making a table or using graphing technology.

Quadratic Relations in Standard Form

The standard form of a quadratic relation is $y = ax^2 + bx + c$. The graph is a parabola. If $a > 0$, the parabola opens upward; if $a < 0$, the parabola opens downward.

EXAMPLE 1 Graphing Using Symmetry

Sketch the graph of $y = -3x^2 - 2x + 7$.

Solution

$y = -3x^2 - 2x + 7$ $\quad$ $a = -3$, so the parabola opens downward.

$y = x(-3x - 2) - 7$ $\quad$ Factor partially.

Let $x = 0$ or $-3x - 2 = 0$ to find two points on the curve.

When $x = 0$, then $y = 7$.

When $-3x - 2 = 0$, then $x = -\frac{2}{3}$ and $y = 7$.

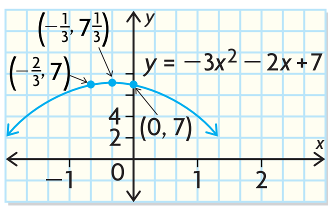
The axis of symmetry is halfway between $(0, 7)$ and $\left(-\frac{2}{3}, 7\right)$.

$$\text{Therefore, } x = \frac{0 + \left(-\frac{2}{3}\right)}{2} = -\frac{1}{3}.$$

To find the vertex, substitute $x = -\frac{1}{3}$ into $y = -3x^2 - 2x + 7$.

$$y = -3\left(-\frac{1}{3}\right)^2 - 2\left(-\frac{1}{3}\right) + 7 = 7\frac{1}{3}$$

The curve opens downward and the vertex is $\left(-\frac{1}{3}, 7\frac{1}{3}\right)$.



Quadratic Relations in Vertex Form

The vertex form of a quadratic relation is $y = a(x - h)^2 + k$, where (h, k) is the vertex. These are also the coordinates of the maximum point when $a < 0$ and of the minimum point when $a > 0$.

EXAMPLE 2 GRAPHING USING THE VERTEX FORM

Sketch the graph of $y = 2(x + 2)^2 + 3$.

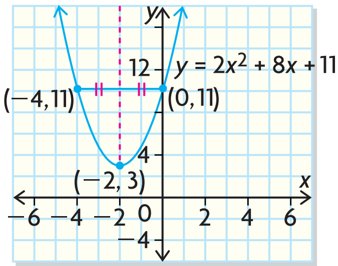
Solution

The equation is in vertex form, and we see that the vertex is $(-2, 3)$. Determine one point on the curve and use symmetry to find a second point.

When $x = 0$,

$$y = 2(0 + 2)^2 + 3$$
$$= 11$$

So, $(0, 11)$ is a point on the curve. Another point, $(-4, 11)$, is symmetric to the axis of symmetry. Now sketch the graph.



In this case, $(-2, 3)$ is the minimum point. The relation has a minimum value of 3 when $x = -2$.

Practising

1. Sketch the graphs, using partial factoring.
   a) $y = 2x^2 - 6x + 5$
   b) $y = -3x^2 + 9x - 2$
   c) $y = 5x^2 - 3 + 5x$
   d) $y = 3 + 4x - 2x^2$
2. Sketch the graphs, using the zeros of the curve.
   a) $y = x^2 + 4x - 12$
   b) $y = x^2 - 7x + 10$
   c) $y = 2x^2 - 5x - 3$
   d) $y = 6x^2 - 13x - 5$
3. Sketch the graphs.
   a) $y = (x - 2)^2 + 3$
   b) $y = (x + 4)^2 - 10$
   c) $y = 2(x - 1)^2 + 3$
   d) $y = -3(x + 1)^2 - 4$

A–13 Completing the Square to Convert to the Vertex Form of a Parabola

A quadratic relation in standard form, $y = ax^2 + bx + c$, can be rewritten in vertex form as $y = a(x - h)^2 + k$ by creating a perfect square in the original and then factoring the square.

Steps Used to Complete the Square	Example: $y = 2x^2 + 12x - 5$
• Divide out the common constant factor from both the x^2- and x-terms.	$y = 2(x^2 + 6x) - 5$
• Determine the constant that must be added (and subtracted) to create a perfect square. This is half the coefficient of the x-term, squared.	$y = 2(x^2 + 6x + 9 - 9) - 5$
• Group the three terms of the perfect square. Multiply the subtracted value and move it outside the bracket.	$y = 2(x^2 + 6x + 9) - 2(9) - 5$
• Factor the perfect square and collect like terms.	$y = 2(x + 3)^2 - 23$

Practising

1. Write each trinomial as a perfect square.

a) $x^2 + 2x + 1$

b) $x^2 + 4x + 4$

c) $x^2 + 6x + 9$

d) $x^2 + 10x + 25$

2. Complete the square, and write in vertex form.

a) $y = x^2 + 2x + 2$

b) $y = x^2 + 4x + 6$

c) $y = x^2 - 12x + 40$

d) $y = x^2 - 18x + 80$

3. Express in vertex form by completing the square. State the equation of the axis of symmetry and the coordinates of the vertex.

a) $y = 2x^2 - 4x + 7$

b) $y = 5x^2 + 10x + 6$

c) $y = -3x^2 - 12x + 2$

d) $y = -2x^2 + 6x + 2$

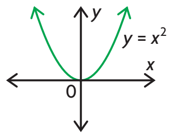
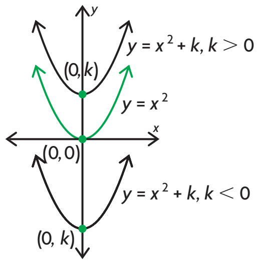
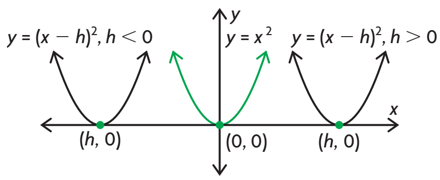
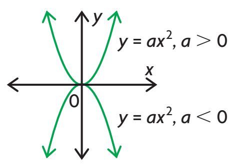
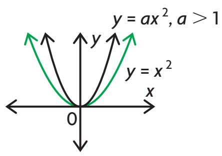
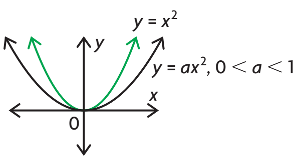
4. A baseball is hit from a height of 1 m. Its height in metres, h, after t seconds is $h = -5t^2 + 10t + 1$.

a) What is the maximum height of the ball?

b) When does the ball reach this height?

A–14 Transformations of Quadratic Relations

The graph of any quadratic relation can be created by altering or repositioning the graph of the base curve, $y = x^2$. To do so, write the relation in vertex form, $y = a(x - h)^2 + k$. The base curve, $y = x^2$, is translated vertically or horizontally, stretched or compressed vertically, or reflected about the x-axis, depending on the values of a, h, and k.

Relation	Type of Transformation	Graph	Explanation
$y = x^2$	Base parabola		This is the base curve upon which other transformations are applied.
$y = x^2 + k$	Vertical Translation The curve shifts up if $k > 0$ and down if $k < 0$.		Add k to the y-coordinate of every point on the base curve. curve. The resultant curve is congruent to the base curve.
$y = (x - h)^2$	Horizontal Translation The curve shifts to the right if $h > 0$ and to the left if $h < 0$.		Subtract h from the x-coordinate of every point on the base curve.
$y = ax^2$	Reflection The curve is reflected about the x-axis if $a < 0$.		Multiply the y-coordinate of every point on the base curve by a.
	Vertical Stretch		The curve has a narrow opening if $a > 1$.
	Vertical Compression		The curve has a wide opening if $0 < a < 1$.

EXAMPLE

Use transformations to graph $y = -2(x + 3)^2 - 4$.

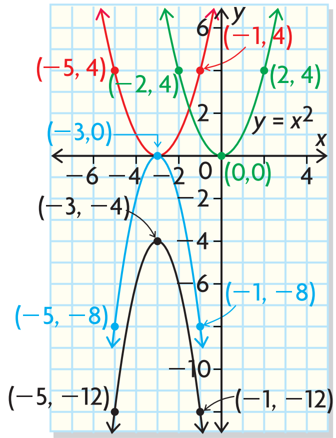
Solution

Step 1: Begin with the graph of the base curve, $y = x^2$ (green). Select three points on the curve to help define its shape. From the form $y = a(x - h)^2 + k$, note that $a = -2$, $h = -3$, and $k = -4$.

Step 2: $h = -3$, so shift the entire parabola 3 units to the left (red).

Step 3: $a = -2$, so multiply every y-coordinate by -2. The parabola is reflected and its opening becomes narrower (blue).

Step 4: $k = -4$, so shift the entire parabola 4 units down. This is the final graph (black).



Practising

In questions 1 and 2, the coordinates of a point on the parabola $y = x^2$ are given. State the new coordinates under each transformation.

1. **a)** (2, 4); shift up 3
b) (−2, 4); shift down 5
c) (−1, 1); shift left 4
d) (1, 1); shift right 6
e) (3, 9); vertical stretch of $-\frac{1}{3}$
f) (2, 4); vertical stretch of 2

2. **a)** (−2, 4); shift left 2 and up 5
b) (−1, 1); shift right 4, vertical stretch 3, and reflect about x-axis
c) (3, 9); vertical compression by 3, shift left 5, and shift down 2
d) (0, 0); shift left 5, vertical stretch 3, shift up 4, and reflect about x-axis

3. Points (−2, 4), (0, 0), and (2, 4) are on the parabola $y = x^2$. Use your knowledge of transformations to determine the equation of the parabola using these coordinates.
a) (−2, 6), (0, 2), (2, 6)
b) (−2, 12), (0, 0), (2, 12)
c) (2, 4), (4, 0), (6, 4)
d) (−2, −6), (0, −2), (2, −6)

In questions 4 and 5, sketch each graph, using transformations on $y = x^2$.

4. **a)** $y = +x^2$
b) $y = -x^2$
c) $y = (x - 3)^2$
d) $y = (x + 4)^2$
e) $y = 2x^2$
f) $y = \frac{1}{2}x^2$

5. **a)** $y = 2(x - 1)^2$
b) $y = -3(x + 1)^2$
c) $y = \frac{1}{2}(x - 2)^2$
d) $y = -\frac{1}{3}(x + 2)^2$

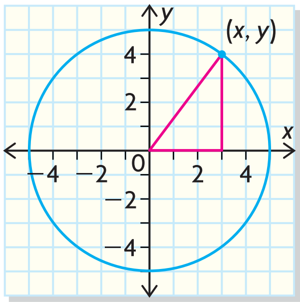
A–15 Equations of Circles Centred at the Origin

A circle can be described by an equation. If the circle is centred at the origin, the equation has a simple form.

Applying the Pythagorean theorem, the coordinates x and y satisfy

$x^2 + y^2 = 5^2$

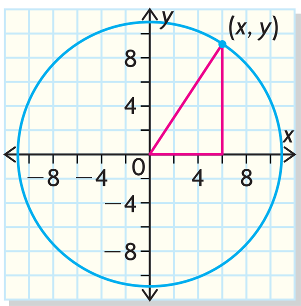
$x^2 + y^2 = 25$



EXAMPLE 1

Write the equation of a circle centred at the origin, with radius 11.

Solution



Applying the Pythagorean theorem,
$x^2 + y^2 = 11^2$
$x^2 + y^2 = 121$

EXAMPLE 2

What is the radius of a circle with equation $x^2 + y^2 = 30$? Give your answer to the nearest hundredth.

Solution

If the radius is r, the equation must be $x^2 + y^2 = r^2$. Therefore,

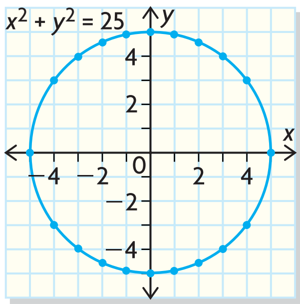
$r^2 = 30$
$r = \sqrt{30}$
$r \doteq 5.48$

Suppose you know the x-coordinate of a point on the circle $x^2 + y^2 = 25$. The possible values of the y-coordinate can be found by substituting for x in the equation. For example, if $x = 3$,

$$
\begin{aligned}
x^2 + y^2 &= 25 \\
(3)^2 + y^2 &= 25 \\
9 + y^2 &= 25 \\
y^2 &= 16 \\
y &= \pm\sqrt{16} = \pm 4
\end{aligned}
$$

The circle $x^2 + y^2 = 25$ can be plotted from a table of values. Since the radius is $\sqrt{25} = 5$, start at $x = -5$ and go through to $x = 5$.

x	−5	−4	−3	−2	−1	0	1	2	3	4	5
y	0	±3	±4	±4.6	±4.9	±5	±4.9	±4.6	±4	±3	0



Notice that there are two y-values for each x-value. The equation of a circle defines a relation that is not a function.

Practising

1. Write the equation of a circle centred at the origin, with each radius.

a) 3
b) 7
c) 8
d) 1

2. What is the radius of a circle with each equation? Round your answer to the nearest hundredth, if necessary.

a) $x^2 + y^2 = 9$
b) $x^2 + y^2 = 81$
c) $x^2 + y^2 = 15$
d) $x^2 + y^2 = 27$
e) $x^2 + y^2 = 6.25$
f) $x^2 + y^2 = 17.64$

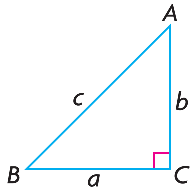
3. A point on the circle $x^2 + y^2 = 169$ has an x-coordinate of 12. What are the possible values of the y-coordinate?

4. Plot the circle with the given equation.

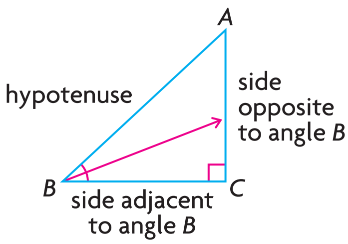
a) $x^2 + y^2 = 16$
b) $x^2 + y^2 = 49$
c) $x^2 + y^2 = 100$
d) $x^2 + y^2 = 12.25$

A–16 Trigonometry of Right Triangles

By the Pythagorean relationship, $a^2 + b^2 = c^2$ for any right triangle, where c is the length of the hypotenuse and a and b are the lengths of the other two sides.



In any right triangle, there are three primary trigonometric ratios that associate the measure of an angle with the ratio of two sides. For example, for $\angle ABC$, in Figure 1,



For $\angle B$

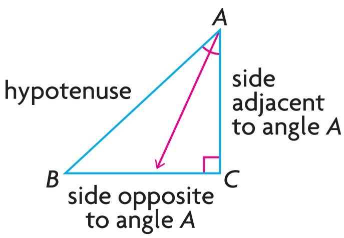
$$\sin B = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos B = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan B = \frac{\text{opposite}}{\text{adjacent}}$$

Figure 1

Similarly, in Figure 2,



For $\angle A$

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}}$$

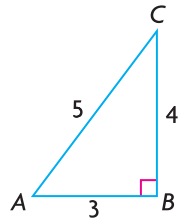
$$\tan A = \frac{\text{opposite}}{\text{adjacent}}$$

Figure 2

Note how the opposite and adjacent sides change in Figures 1 and 2 with angles A and B.

EXAMPLE 1

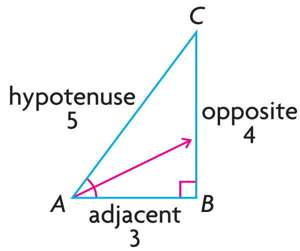
State the primary trigonometric ratios of $\angle A$.



Solution

Sketch the triangle. Then label the opposite side, the adjacent side, and the hypotenuse.

$$\sin A = \frac{\text{opposite}}{\text{hypotenuse}}$$
$$= \frac{4}{5}$$



$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}}$$
$$= \frac{3}{5}$$

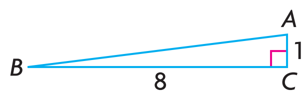
$$\tan A = \frac{\text{opposite}}{\text{adjacent}}$$
$$= \frac{4}{3}$$

EXAMPLE 2

A ramp must have a rise of one unit for every eight units of run. What is the angle of inclination of the ramp?

Solution

The slope of the ramp is $\frac{\text{rise}}{\text{run}} = \frac{1}{8}$. Draw a labelled sketch.



Calculate the measure of $\angle B$ to determine the angle of inclination.

The trigonometric ratio that associates $\angle B$ with the opposite and adjacent sides is the tangent. Therefore,

$$\tan B = \frac{\text{opposite}}{\text{adjacent}}$$

$$\tan B = \frac{1}{8}$$

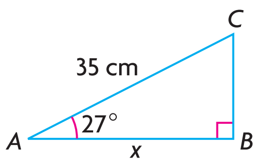
$$B = \tan^{-1}\left(\frac{1}{8}\right)$$

$$B \doteq 7°$$

The angle of inclination is about 7°.

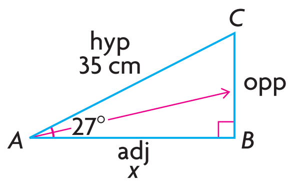
EXAMPLE 3

Determine x to the nearest centimetre.



Solution

Label the sketch. The cosine ratio associates $\angle A$ with the adjacent side and the hypotenuse.



Then,

$$\cos A = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\cos 27° = \frac{x}{35}$$

$$x = 35 \cos 27° \doteq 31$$

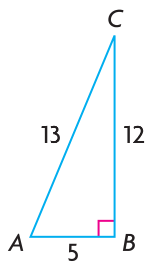
So x is about 31 cm.

Practising

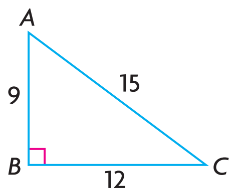
1. A rectangular lot is 15 m by 22 m. How long is the diagonal, to the nearest metre?

2. State the primary trigonometric ratios for $\angle A$.

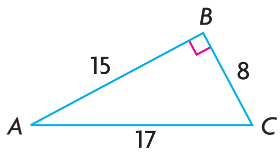
a)



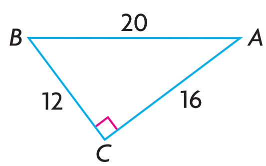
b)



c)



d)



3. Solve for x to one decimal place.

a) $\sin 39° = \frac{x}{7}$

b) $\cos 65° = \frac{x}{16}$

c) $\tan 15° = \frac{x}{22}$

d) $\tan 49° = \frac{31}{x}$

4. Solve for $\angle A$ to the nearest degree.

a) $\sin A = \frac{5}{8}$

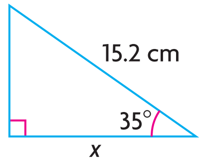
b) $\cos A = \frac{13}{22}$

c) $\tan B = \frac{19}{22}$

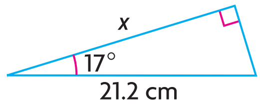
d) $\cos B = \frac{3}{7}$

5. Determine x to one decimal place.

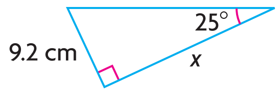
a)



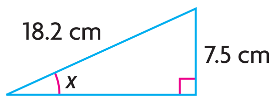
b)



c)



d)



6. In $\triangle ABC$, $\angle B = 90°$ and $AC = 13$ cm. Determine

a) BC if $\angle C = 17°$

b) AB if $\angle C = 26°$

c) $\angle A$ if $BC = 6$ cm

d) $\angle C$ if $BC = 9$ cm

7. A tree casts a shadow 9.3 m long when the angle of the sun is 43°. How tall is the tree?

8. Janine stands 30.0 m from the base of a communications tower. The angle of elevation from her eyes to the top of the tower is 70°. How high is the tower if her eyes are 1.8 m above the ground?

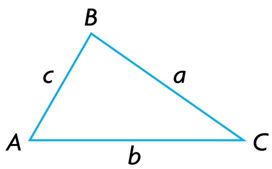
9. A surveillance camera is mounted on the top of a building that is 80 m tall. The angle of elevation from the camera to the top of another building is 42°. The angle of depression from the camera to the same building is 32°. How tall is the other building?

A–17 Trigonometry of Acute Triangles: The Sine Law and the Cosine Law

An acute triangle contains three angles less than 90°.

Sine Law

The sine law states that, for $\triangle ABC$,

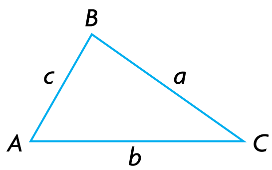


- $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$ or
- $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

To use the sine law, two angles and one side (AAS) or two sides and an opposite angle (SSA) must be given.

Cosine Law

The cosine law states that, for $\triangle ABC$,



- $c^2 = a^2 + b^2 - 2ab \cos C$ or
- $a^2 = b^2 + c^2 - 2bc \cos A$ or
- $b^2 = a^2 + c^2 - 2ac \cos B$

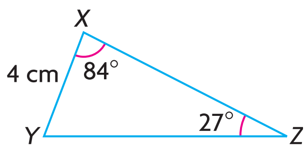
To use the cosine law, two sides and the contained angle (SAS) or three sides (SSS) must be given.

EXAMPLE 1

Determine the length of XZ to one decimal place.

Solution

In the triangle shown, $\angle X = 84°$, $\angle Z = 27°$, and $z = 4$ cm.



$$\begin{aligned}\angle Y &= 180° - (84° + 27°)\\ &= 180° - 111°\\ &= 69°\end{aligned}$$

This is not a right triangle, so the primary trigonometric ratios do not apply. Two angles and one side are known, so the sine law can be used.

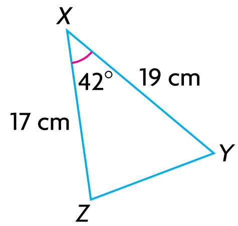
$$\begin{aligned}\frac{\sin Y}{y} &= \frac{\sin Z}{z}\\ \frac{\sin 69°}{y} &= \frac{\sin 27°}{4}\\ y &= \frac{4 \sin 69°}{\sin 27°}\\ &\doteq 8.2\end{aligned}$$

Then XZ is about 8.2 cm.

EXAMPLE 2

Determine the length of ZY to one decimal place.

Solution



In this triangle, $\angle X = 42°$, $y = 17$ cm, and $z = 19$ cm.

There is not enough information to use the primary trigonometric ratios or the sine law. However, two sides and the contained angle are known, so the cosine law can be used.

$$x^2 = y^2 + z^2 - 2yz \cos X$$

$$x^2 = 17^2 + 19^2 - 2(17)(19) \cos 42°$$

$$x = \sqrt{17^2 + 19^2 - 2(17)(19) \cos 42°}$$

$$x \doteq 13$$

Therefore, ZY is about 13 cm.

Practising

1. Solve to one decimal place.

a) $\dfrac{\sin 35°}{c} = \dfrac{\sin 42°}{12}$

b) $\dfrac{15}{\sin 43°} = \dfrac{13}{\sin B}$

c) $19^2 = 15^2 + 13^2 - 2(15)(13)\cos A$

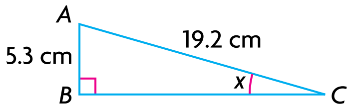
d) $c^2 = 12^2 + 17^2 - 2(12)(17)\cos 47°$

e) $\dfrac{\sin A}{12.3} = \dfrac{\sin 58°}{14.2}$

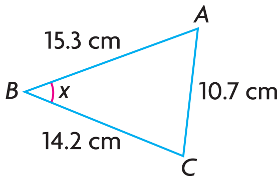
f) $\dfrac{\sin 14°}{3.1} = \dfrac{\sin 27°}{b}$

2. Determine x to one decimal place.

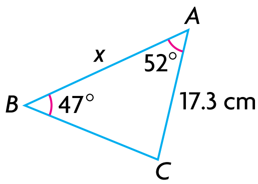
a)



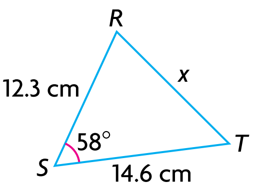
b)



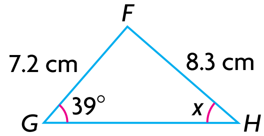
c)



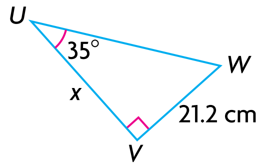
d)



e)



f)



3. Solve each triangle for all missing sides and angles.

a) $\triangle CAT$, with $c = 5.2$ cm, $a = 6.8$ cm, and $\angle T = 59°$.

b) $\triangle ABC$, with $a = 4.3$ cm, $b = 5.2$ cm, and $c = 7.5$.

c) $\triangle DEF$, with $DE = 14.3$ cm, $EF = 17.2$ cm, and $\angle D = 39°$.

4. A swamp separates points L and R. To determine the distance between them, Ciana stands at L and looks toward R. She turns about 45° and walks 52 paces from L to point P. From P, she looks at R and estimates that $\angle LPR$ is about 60°. How many paces is it from L to R?

5. An observation helicopter using a laser device determines that the helicopter is 1800 m from a boat in distress. The helicopter is 1200 m from a rescue boat. The angle formed between the helicopter and the two boats is 35°. How far apart are the boats?

6. Neil designs a cottage that is 15 m wide. The roof rafters are the same length and meet at an angle of 80°. The rafters hang over the supporting wall by 0.5 m. How long are the rafters?