8.5 Annuities: Present Value

GOAL

Determine the present value of an annuity earning compound interest.

YOU WILL NEED

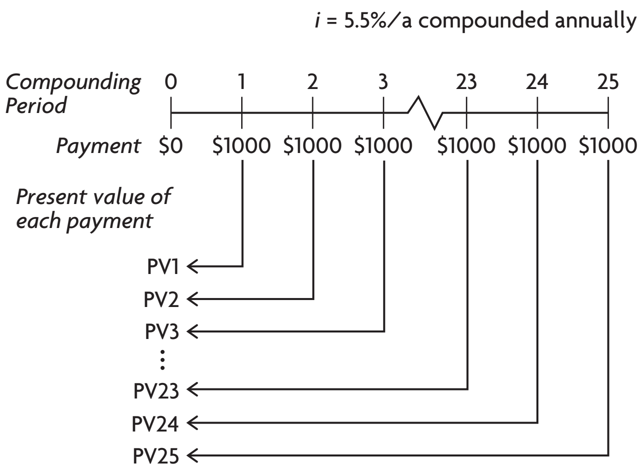
- graphing calculator
- spreadsheet software

INVESTIGATE the Math

Kew wants to invest some money at 5.5%/a compounded annually. He would like the investment to provide $1000 for scholarships at his old high school at the end of each year for the next 25 years.

? How much should Kew invest now?

A. Copy the timeline shown. How would you calculate each of the present values PV1 to PV25?

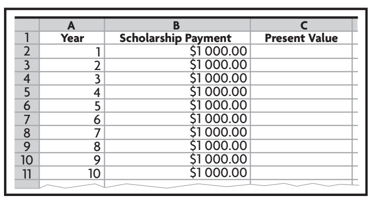


B. How much would Kew need to invest now if he wanted to provide $1000 at the end of the 1st year?

C. How much would Kew need to invest now if he wanted to provide $1000 at the end of the 2nd, 3rd, and 4th years, respectively?

D. How is the present value after 2 years (PV2) related to the present value after 1 year (PV1)?

E. Set up a spreadsheet with columns as shown at the right. Enter your values of PV1 and PV2 in the Present Value column.



	A	B	C
1	Year	Scholarship Payment	Present Value
2	1	$1 000.00	
3	2	$1 000.00	
4	3	$1 000.00	
5	4	$1 000.00	
6	5	$1 000.00	
7	6	$1 000.00	
8	7	$1 000.00	
9	8	$1 000.00	
10	9	$1 000.00	
11	10	$1 000.00	

F. Use the relationship among the present values to complete the rest of the entries under Present Value.

G. Use the values in the Present Value column to determine how much Kew would need to invest now in order to provide the scholarships for the next 25 years.

Reflecting

H. What type of sequence do the present values in part F form?

I. Describe a method that you could have used to solve this problem without using a spreadsheet.

APPLY the Math

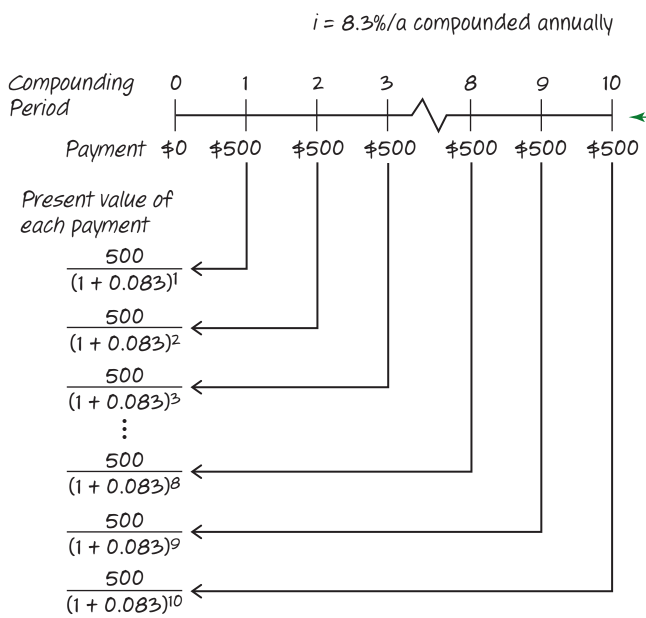
EXAMPLE 1 Representing the present value of an annuity earning compound interest as a series

a) How much would you need to invest now at 8.3%/a compounded annually to provide \$500 per year for the next 10 years?

b) How much would you need to invest now to provide n regular payments of \$$R$ if the money is invested at a rate of i% per compounding period?

Tara's Solution

a)



I drew a timeline showing the \$500 payments for the next 10 years.

$$PV = \frac{A}{(1 + i)^n}$$

$$PV_1 = \frac{500}{(1.083)}$$

$$PV_2 = \frac{500}{(1.083)^2}$$

$$PV_3 = \frac{500}{(1.083)^3}$$

$$\vdots$$

I considered each payment separately and used the present-value formula to determine how much would need to be invested now to provide each \$500 payment.

$$PV_9 = \frac{500}{(1.083)^9}$$

$$PV_{10} = \frac{500}{(1.083)^{10}}$$

$$a = \frac{500}{(1.083)} = 500 \times 1.083^{-1}$$

The present values for each payment are the first 10 terms of a geometric sequence with first term 500×1.083^{-1} and common ratio 1.083^{-1}.

$$r = \frac{1}{1.083} = 1.083^{-1}$$

$$n = 10$$

$$S_{10} = 500 \times 1.083^{-1} + 500 \times 1.083^{-2} + 500 \times 1.083^{-3} + \ldots + 500 \times 1.083^{-9} + 500 \times 1.083^{-10}$$

The total amount of money invested now has to provide each of the $500 future payments. So I had to calculate the sum of all of the present values.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

The sum of the present values forms a geometric series, so I used the formula for the sum of a geometric series.

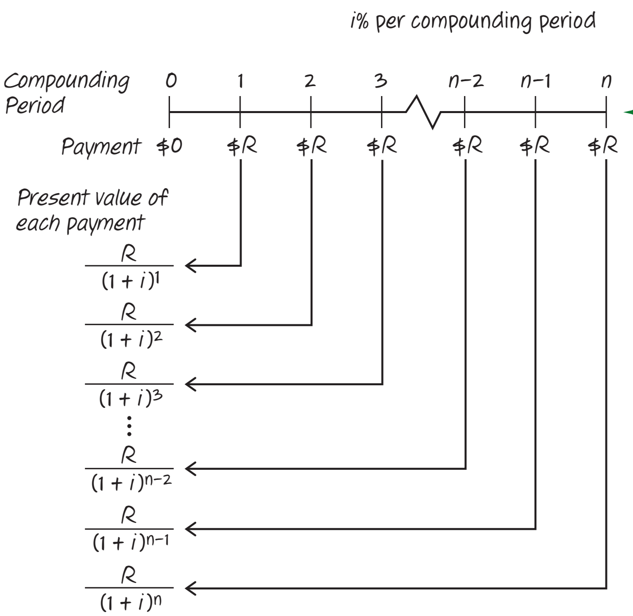
$$S_{10} = \frac{500 \times 1.083^{-1}[(1.083^{-1})^{10} - 1]}{1.083^{-1} - 1}$$

$$\doteq \$3310.11$$

I rounded to the nearest cent.

A sum of $3310.11 invested now would provide a payment of $500 for each of the next 10 years.

b)



I drew a timeline showing the $$R$ payments for n regular intervals.

$$PV = \frac{A}{(1 + i)^n}$$

I considered each payment separately and used the present-value formula to determine how much would need to be invested now to provide each specific $R payment.

$$PV_1 = \frac{R}{1 + i}$$

$$PV_2 = \frac{R}{(1 + i)^2}$$

$$PV_3 = \frac{R}{(1 + i)^3}$$

$$\vdots$$

$$PV_n = \frac{R}{(1 + i)^n}$$

$$a = R \times (1 + i)^{-1}$$

$$r = (1 + i)^{-1}$$

I used negative exponents, since I was dividing by $1 + i$ each time.

The present values for each payment are the first n terms of a geometric sequence with first term $R \times (1 + i)^{-1}$ and common ratio $(1 + i)^{-1}$.

$$S_n = R \times (1 + i)^{-1} + R \times (1 + i)^{-2} + R \times (1 + i)^{-3} + \ldots + R \times (1 + i)^{-n}$$

I needed to determine the total amount of money invested now to provide each of the $R future payments. So I had to calculate the sum of all of the present values.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

The sum of the present values forms a geometric series, so I used the formula for the sum of a geometric series.

$$= \frac{R \times (1 + i)^{-1}[(1 + i)^{-n} - 1]}{(1 + i)^{-1} - 1}$$

$$= \frac{R \times (1 + i)^{-1}[(1 + i)^{-n} - 1]}{(1 + i)^{-1} - 1} \times \frac{1 + i}{1 + i}$$

The numerator and denominator each have a factor of $(1 + i)^{-1}$, so I multiplied them both by $1 + i$ to simplify.

$$= \frac{R[(1 + i)^{-n} - 1]}{1 - (1 + i)}$$

$$= \frac{R[(1 + i)^{-n} - 1]}{-i}$$

$$= R \times \left(\frac{1 - (1 + i)^{-n}}{i}\right)$$

I multiplied the numerator and denominator by -1 to simplify.

The present value of an annuity in which $R is paid at the end of each of n regular intervals earning i% compound interest per interval is $PV = R \times \left(\frac{1 - (1 + i)^{-n}}{i}\right)$.

EXAMPLE 2 Selecting a strategy to determine the present value of an annuity

Sharon won a lottery that offers $50 000 a year for 20 years or a lump-sum payment now. If she can invest the money at 5%/a compounded annually, how much should the lump-sum payment be to be worth the same amount as the annuity?

Joel's Solution: Using a Spreadsheet

	A	B	C
1	Year	Payment	Present Value
2	1	$50 000.00	"= B2/1.05"
3	2	$50 000.00	"= B3/(1.05)^A3"
4	3	$50 000.00	"= B4/(1.05)^A4"

I set up a spreadsheet to determine the present value of each of the payments for the next 20 years. The present value of each payment is given by the formula $PV = \frac{A}{(1 + i)^n}$, so the present value of the payments form a geometric sequence with $r = \frac{1}{1 + i}$. Since Sharon is earning 5%/a, the present value of each following year is equal to 1.05 times the present value of the previous year.

	A	B	C
1	Year	Payment	Present Value
2	1	$50 000.00	$47 619.05
3	2	$50 000.00	$45 351.47
4	3	$50 000.00	$43 191.88
5	4	$50 000.00	$41 135.12
6	5	$50 000.00	$39 176.31
7	6	$50 000.00	$37 310.77
8	7	$50 000.00	$35 534.07
9	8	$50 000.00	$33 841.97
10	9	$50 000.00	$32 230.45
11	10	$50 000.00	$30 695.66
12	11	$50 000.00	$29 233.96
13	12	$50 000.00	$27 841.87
14	13	$50 000.00	$26 516.07
15	14	$50 000.00	$25 253.40
16	15	$50 000.00	$24 050.85
17	16	$50 000.00	$22 905.58
18	17	$50 000.00	$21 814.83
19	18	$50 000.00	$20 776.03
20	19	$50 000.00	$19 786.70
21	20	$50 000.00	$18 844.47
22			$623 110.52

I used the FILL DOWN command to determine the present values for the remaining payments. I then used the SUM command to determine the sum of all the present values.

The lump-sum payment should be $623 110.52.

EXAMPLE 3 Selecting a strategy to determine the regular payment and total interest of an annuity

Len borrowed \$200 000 from the bank to purchase a yacht. If the bank charges 6.6%/a compounded monthly, he will take 20 years to pay off the loan.

a) How much will each monthly payment be?
b) How much interest will he have paid over the term of the loan?

Jasmine's Solution: Using the Formula

a)

$$i = \frac{0.066}{12} = 0.0055$$

$$n = 20 \times 12 = 240$$

$$PV = \$200\,000$$

I calculated the interest rate per compounding period and the number of compounding periods.

$$PV = R \times \left(\frac{1 - (1 + i)^{-n}}{i}\right)$$

$$200\,000 = R \times \left(\frac{1 - (1 + 0.0055)^{-240}}{0.0055}\right)$$

I substituted the values of *PV*, *i*, and *n* into the formula for the present value of an annuity.

$$200\,000 \doteq R \times 133.072$$

$$\frac{200\,000}{133.072} = R \times \frac{133.072}{133.072}$$

To solve for *R*, I divided both sides of the equation by 133.072.

$$R \doteq 1502.94$$

I rounded to the nearest cent.

Len will have to pay \$1502.94 per month for 20 years to pay off the loan.

b)

$$A = 1502.94 \times 240$$
$$= \$360\,706.60$$

I calculated the total amount that Len will have paid over the 20-year term.

$$I = A - PV$$
$$= \$360\,706.60 - \$200\,000$$
$$= \$160\,706.60$$

I determined the interest by subtracting the present value from the total amount that Len will have paid.

Over the 20-year term of the loan, Len will have paid \$160 706.60 in interest.

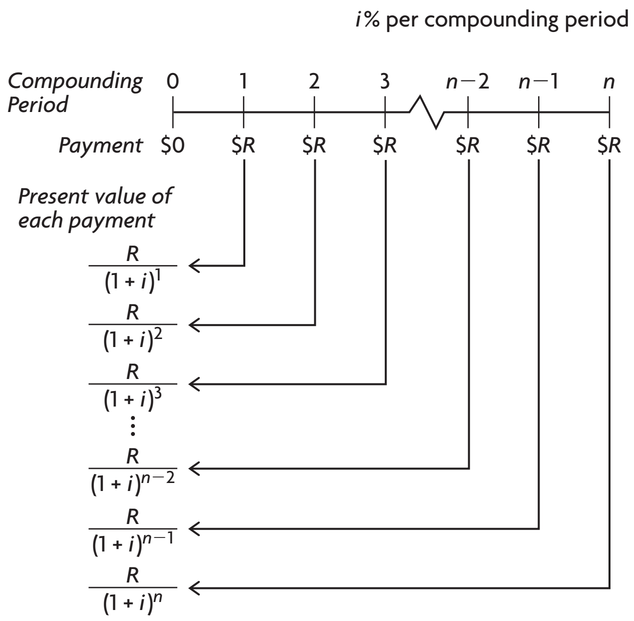
In Summary

Key Ideas

- The present value of an annuity is the value of the annuity at the beginning of the term. It is the sum of all present values of the payments and can be written as the geometric series

$$PV = R \times (1 + i)^{-1} + R \times (1 + i)^{-2} + R \times (1 + i)^{-3} + \ldots + R \times (1 + i)^{-n}$$

where PV is the present value; R is the regular payment; i is the interest rate per compounding period, expressed as a decimal; and n is the number of compounding periods.



- The formula for the sum of a geometric series can be used to determine the present value of an annuity.

Need to Know

- The formula for the present value of an annuity is

$$PV = R \times \left(\frac{1 - (1 + i)^{-n}}{i}\right)$$

where PV is the present value; R is the regular payment each compounding period; i is the interest rate per compounding period, expressed as a decimal; and n is the number of compounding periods.

CHECK Your Understanding

1. Each situation represents a loan.

i) Draw a timeline to represent the amount of the original loan.
ii) Write the series that represents the amount of the original loan.
iii) Calculate the amount of the original loan.
iv) Calculate the interest paid.

	Regular Payment	Rate of Compound Interest per Year	Compounding Period	Time
a)	\$650 per year	3.7%	annually	5 years
b)	\$1200 every 6 months	9.4%	semi-annually	9 years
c)	\$84.73 per quarter	3.6%	quarterly	$3\frac{1}{2}$ years
d)	\$183.17 per month	6.6%	monthly	10 years

2. Each situation represents a simple, ordinary annuity.

i) Calculate the present value of each payment.
ii) Write the present values of the payments as a series.
iii) Calculate the present value of the annuity.

	Regular Payment	Rate of Compound Interest per Year	Compounding Period	Time
a)	\$8000 per year	9%	annually	7 years
b)	\$300 every 6 months	8%	semi-annually	3.5 years
c)	\$750 per quarter	8%	quarterly	2 years

PRACTISING

3. Calculate the present value of each annuity.
K

	Regular Payment	Rate of Compound Interest per Year	Compounding Period	Time
a)	\$5000 per year	7.2%	annually	5 years
b)	\$250 every 6 months	4.8%	semi-annually	12 years
c)	\$25.50 per week	5.2%	weekly	100 weeks
d)	\$48.50 per month	23.4%	monthly	$2\frac{1}{2}$ years

4. You want to buy a \$1300 stereo on credit and make monthly payments over 2 years. If the store is charging you 18%/a compounded monthly, what will be your monthly payments?

5. Lily wants to buy a snowmobile. She can borrow \$7500 at 10%/a compounded quarterly if she repays the loan by making equal quarterly payments for 4 years.

a) Draw a timeline to represent the annuity.

b) Write the series that represents the present value of the annuity.

c) Calculate the quarterly payment that Lily must make.

6. Rocco pays \$50 for a DVD/CD player and borrows the remaining amount. He plans to make 10 monthly payments of \$40 each. The first payment is due next month.

a) The interest rate is 18%/a compounded monthly. What was the selling price of the player?

b) How much interest will he have paid over the term of the loan?

7. Emily is investing \$128 000 at 7.8%/a compounded monthly. She wants to withdraw an equal amount from this investment each month for the next 25 years as spending money. What is the most she can take out each month?

8. The Peca family wants to buy a cottage for \$69 000. The Pecas can pay \$5000 and finance the remaining amount with a loan at 9%/a compounded monthly. The loan payments are monthly, and they may choose either a 7-year or a 10-year term.

a) Calculate the monthly payment for each term.

b) How much would they save in interest by choosing the shorter term?

c) What other factors should the Pecas consider before making their financing decision?

9. A Charles would like to buy a new car that costs \$32 000. The dealership offers to finance the car at 2.4%/a compounded monthly for five years with monthly payments. The dealer will reduce the selling price by \$3000 if Charles pays cash. Charles can get a loan from his bank at 5.4%/a compounded monthly. Which is the best way to buy the car? Justify your answer with calculations.

10. To pay off \$35 000 in loans, Nina's bank offers her a rate of 8.4%/a compounded monthly. She has a choice between a 5-, 10-, or 15-year term.

a) Determine the monthly payment for each term.

b) Calculate how much interest Nina would pay in each case.

11. Pedro pays \$45 for a portable stereo and borrows the remaining amount. The loan payments are \$25 per month for 1 year. The interest rate is 18.6%/a compounded monthly.

a) What was the selling price of the stereo?

b) How much interest will Pedro have paid over the term of the loan?

12. Suzie buys a new computer for \$2500. She pays \$700 and finances the rest at \$75.84 per month for $2\frac{1}{2}$ years. What annual interest rate, compounded monthly, is Suzie being charged? Round your answer to two decimal places.

13. Leo invests \$50 000 at 11.2%/a compounded quarterly for his retirement. Leo's financial advisor tells him that he should take out a regular amount quarterly when he retires. If Leo has 20 years until he retires and wants to use the investment for recreation for the first 10 years of retirement, what is the maximum quarterly withdrawal he can make?

14. Charmaine calculates that she will require about \$2500 per month for the first 15 years of her retirement. If she has 25 years until she retires, how much should she invest each month at 9%/a compounded monthly for the next 25 years if she plans to withdraw \$2500 per month for the 15 years after that?

15. **T** A lottery has two options for winners collecting their prize:
- Option A: \$1000 each week for life
- Option B: \$660 000 in one lump sum

The current interest rate is 6.76%/a compounded weekly.

a) Which option would you suggest to a winner who expects to live for another 25 years?

b) When is option A better than option B?

16. **C** Classify situations and factors that show the differences between each pair of terms. Give examples.

a) a lump sum or an annuity

b) future value or present value

Extending

17. Stefan claims that he has found a different method for calculating the present value of an annuity. Instead of calculating the present value of each payment, he calculates the future value of each payment. Then he calculates the sum of the future values of the payments. Finally, he calculates the present value of this total sum.

a) Use Stefan's method to solve Example 1 (a).

b) Create another example to show that his claim is true. Include timelines.

c) Use the formula for present value to prove that Stefan's claim works for all annuities.

18. Kyla must repay student loans that total \$17 000. She can afford to make \$325 monthly payments. The bank is charging an interest rate of 7.2%/a compounded monthly. How long will it take Kyla to repay her loans?

19. In question 14, Charmaine invested a fixed amount per month so that her annuity would provide her with another monthly amount in her retirement. Derive a formula for the regular payment $\$R$ that must be made for m payments at an interest rate of i% per compounding period to provide for a regular withdrawal of $\$W$ after all the payments are made for n withdrawals.