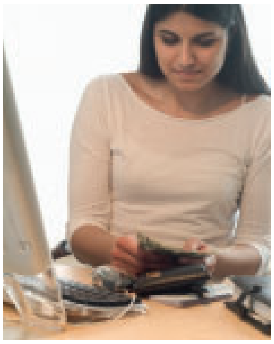
8.4 Annuities: Future Value

YOU WILL NEED

- graphing calculator
- spreadsheet software



GOAL

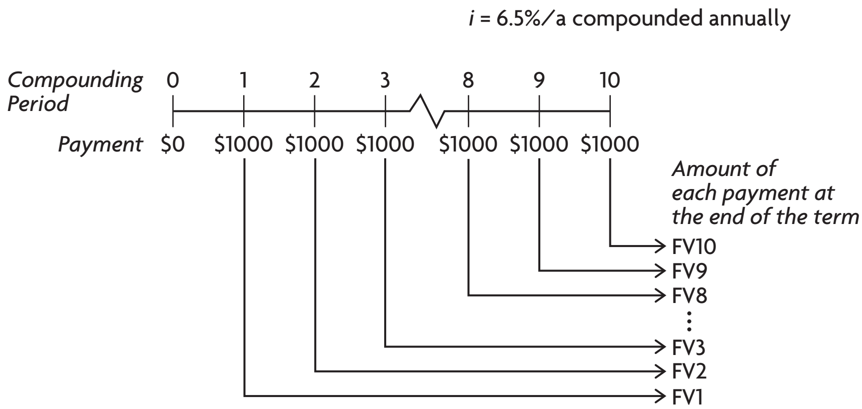
Determine the future value of an annuity earning compound interest.

INVESTIGATE the Math

Christine decides to invest $1000 at the end of each year in a Canada Savings Bond earning 6.5%/a compounded annually. Her first deposit is on December 31, 2007.

? How much will her investments be worth 10 years later, on January 1, 2017?

A. Copy the timeline shown. How would you calculate each of the future values FV1 to FV10?



B. Set up a spreadsheet with columns as shown. Copy the data already entered. Complete the entries under Date Invested up to Dec. 31, 2007.

	A	B	C	D
1	Date Invested	Amount Invested	Number of Years Invested	Value on Jan. 1, 2017
2	Dec. 31, 2016	$1 000.00	0	$1 000.00
3	Dec. 31, 2015	$1 000.00	1	
4	Dec. 31, 2014	$1 000.00	2	

C. Fill in cells D3 and D4 to show what the investments made on Dec. 31, 2015, and Dec. 31, 2014, respectively will be worth on Jan. 1, 2017.

D. How is the value in cell D3 (FV9) related to the value in cell D2 (FV10)? How is the value in cell D4 (FV8) related to the value in cell D3 (FV9)?

E. Use the pattern from part D to complete the rest of the entries under Value on Jan. 1, 2017.

F. What type of sequence do the values on Jan. 1, 2017 form?

G. Calculate the total amount of all the investments at the end of 10 years for this **annuity**.

annuity

a series of payments or investments made at regular intervals. A **simple** annuity is an annuity in which the payments coincide with the compounding period, or *conversion* period. An **ordinary** annuity is an annuity in which the payments are made at the end of each interval. Unless otherwise stated, each annuity in this chapter is a simple, ordinary annuity.

Reflecting

H. The values of all of the investments at the end of each year for 10 years formed a specific type of sequence. How is the total value of the annuity at the end of 10 years related to a series?

I. How could you use the related series to solve problems involving annuities?

APPLY the Math

EXAMPLE 1 Representing the future value of an annuity earning compound interest as a series

a) Hans plans to invest \$1000 at the end of each 6-month period in an annuity that earns 4.8%/a compounded semi-annually for the next 20 years. What will be the future value of his annuity?

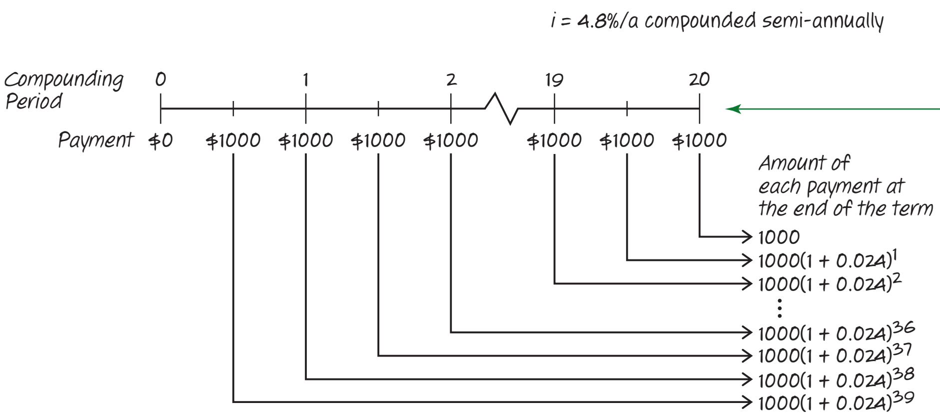
b) You plan to invest $\$R$ at regular intervals in an annuity that earns i% compounded at the end of each interval. What will be the future value, FV, of your annuity after n intervals?

Barbara's Solution

a) $i = \frac{0.048}{2} = 0.024$

$n = 20 \times 2 = 40$

Since the interest is paid semi-annually, I calculated the interest rate per compounding period and the number of compounding periods.



I drew a timeline of the investments for each compounding period, and I represented the amount of each investment.

The last \$1000 investment earned no interest because it was deposited at the end of the term.

The first \$1000 investment earned interest over 39 periods. It didn't earn interest during the first compounding period because it was deposited at the end of that period.

$$1000, 1000(1.024), 1000(1.024)^2, \ldots , 1000(1.024)^{38}, 1000(1.024)^{39}$$

The future values of all of the investments form a geometric sequence with first term \$1000 and common ratio 1.024.

$$S_{40} = 1000 + 1000(1.024) + 1000(1.024)^2 + \ldots + 1000(1.024)^{38} + 1000(1.024)^{39}$$

The total amount of all these investments is the first 40 terms of a geometric series.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{40} = \frac{1000(1.024^{40} - 1)}{1.024 - 1}$$

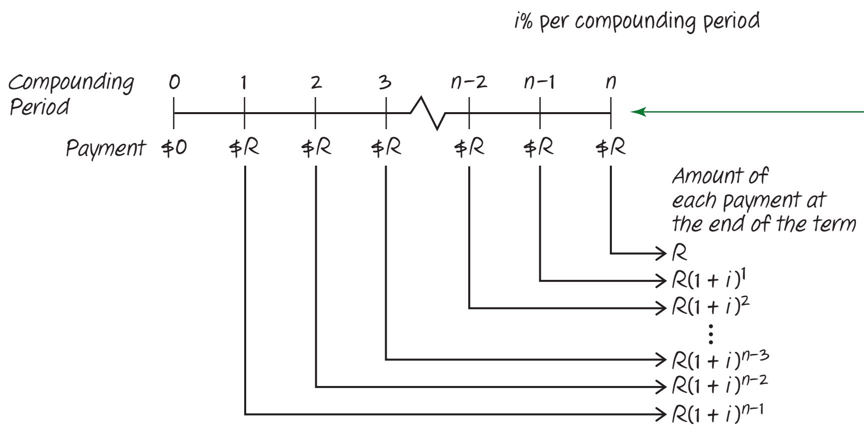
I used the formula for the sum of a geometric series to calculate the total amount of all of Hans's investments.

$$\doteq \$65\ 927.08$$

I rounded to the nearest cent.

The future value of Hans's annuity at the end of 20 years is \$65 927.08.

b)



I drew a timeline of the investments for each compounding period to show the amount of each investment.

The last \$$R$ investment earned no interest. The first \$$R$ investment earned interest $n - 1$ times.

$$R, R(1 + i), R(1 + i)^2, \ldots , R(1 + i)^{n-2}, R(1 + i)^{n-1}$$

The values of all of the investments form a geometric sequence with first term R and common ratio $1 + i$.

$$S_n = R + R(1 + i) + R(1 + i)^2 + \ldots + R(1 + i)^{n-2} + R(1 + i)^{n-1}$$

The total amount of all these investments is the first n terms of a geometric series.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$= \frac{R[(1 + i)^n - 1]}{(1 + i) - 1}$$

$$= R \times \left(\frac{(1 + i)^n - 1}{i}\right)$$

I used the formula for the sum of a geometric series to calculate the total amount of all the investments.

The future value of an annuity in which \$$R$ is invested at the end of each of n regular intervals earning i% of compound interest per interval is

$$FV = R \times \left(\frac{(1 + i)^n - 1}{i}\right),$$ where i is expressed as a decimal.

EXAMPLE 2 Selecting a strategy to determine the future value of an annuity

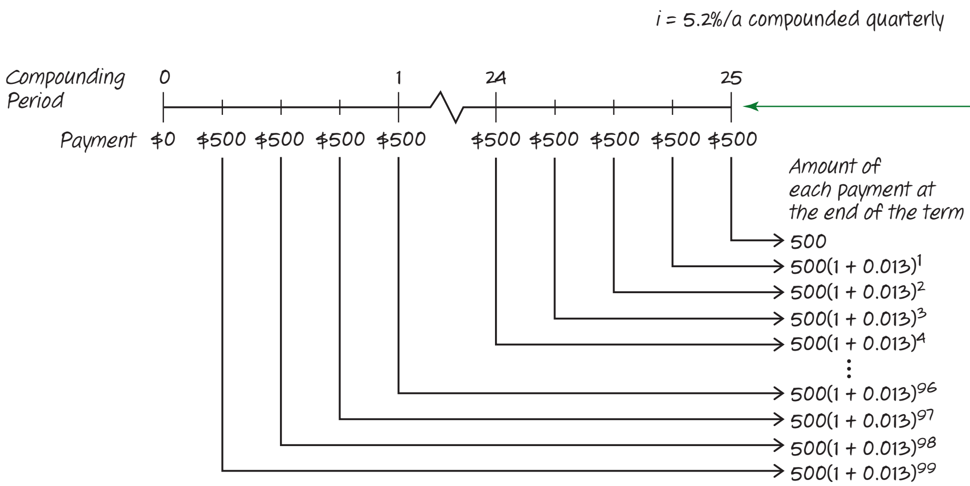
Chie puts away $500 every 3 months at 5.2%/a compounded quarterly. How much will her annuity be worth in 25 years?

Kew's Solution: Using a Geometric Series

$$i = \frac{0.052}{4} = 0.013$$

$$n = 25 \times 4 = 100$$

First I calculated the interest rate per compounding period and the number of compounding periods.



I drew a timeline of the investments for each quarter to show the amounts of each investment. I calculated the value of each investment at the end of 25 years.

$$500,\ 500(1.013),\ 500(1.013)^2,\ \ldots,\ 500(1.013)^{98},\ 500(1.013)^{99}$$

The values form a geometric sequence with first term $500 and common ratio 1.013.

$$S_{100} = 500 + 500(1.013) + 500(1.013)^2 + \ldots + 500(1.013)^{98} + 500(1.013)^{99}$$

The total amount of all these investments is the first 100 terms of a geometric series.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{100} = \frac{500(1.013^{100} - 1)}{1.013 - 1}$$

I used the formula for the sum of a geometric series to calculate the total amount of all of Chie's investments.

$$\doteq \$101\ 487.91$$

I rounded to the nearest cent.

The total amount of all of Chie's investments at the end of 25 years will be $101 487.91.

Tina's Solution: Using the Formula for the Future Value of an Annuity

$R = \$500$

$i = \frac{0.052}{4} = 0.013$ ← I calculated the interest rate per compounding period and the number of compounding periods.

$n = 25 \times 4 = 100$

$FV = R \times \left(\frac{(1 + i)^n - 1}{i}\right)$ ← I substituted the values of R, i, and n into the formula for the future value of a simple, ordinary annuity.

$= 500 \times \left(\frac{(1 + 0.013)^{100} - 1}{0.013}\right)$

$\doteq \$101\ 487.91$ ← I rounded to the nearest cent.

The future value of Chie's annuity will be \$101 487.91.

EXAMPLE 3 Selecting a strategy to determine the regular payment of an annuity

Sam wants to make monthly deposits into an account that guarantees 9.6%/a compounded monthly. He would like to have \$500 000 in the account at the end of 30 years. How much should he deposit each month?

Chantal's Solution

$i = \frac{0.096}{12} = 0.008$ ← I calculated the interest rate per compounding period and the number of compounding periods.

$n = 30 \times 12 = 360$

$FV = \$500\ 000$ ← The future value of the annuity is \$500 000.

$FV = R \times \left(\frac{(1 + i)^n - 1}{i}\right)$

$500\ 000 = R \times \left(\frac{(1 + 0.008)^{360} - 1}{0.008}\right)$ ← I substituted the values of FV, i, and n into the formula for the future value of an annuity.

$$500\ 000 \doteq R \times 2076.413$$

$$\frac{500\ 000}{2076.413} = R \times \frac{2076.413}{2076.413}$$

To solve for R, I divided both sides of the equation by 2076.413.

$$R = \$240.80$$

I rounded to the nearest cent.

Sam would have to deposit $240.80 into the account each month in order to have $500 000 at the end of 30 years.

Tech Support

For help using a spreadsheet to enter values and formulas, and to fill down, see Technical Appendix, B-21.

EXAMPLE 4 Selecting a strategy to determine the term of an annuity

Nahid borrows $95 000 to buy a cottage. She agrees to repay the loan by making equal monthly payments of $750 until the balance is paid off. If Nahid is being charged 5.4%/a compounded monthly, how long will it take her to pay off the loan?

Zak's Solution

	A	B	C	D	E
1	Payment Number	Payment	Interest Paid	Principal Paid	Balance
2					$95 000.00
3	1	$750.00	$427.50	$322.50	$94 677.50
4	2	$750.00	"=E3*0.054/12"	"=B4-C4"	"=E3-D4"

I set up a spreadsheet to calculate the balance after every payment. The interest is always charged on the balance and is $\frac{1}{12}$ of 5.4% since it is compounded monthly. The part of the principal that is paid off with each payment is $750, less the interest. The new balance is the old balance, less the part of the principal that is paid.

	A	B	C	D	E
1	Payment Number	Payment	Interest Paid	Principal Paid	Balance
2					$95 000.00
3	1	$750.00	$427.50	$322.50	$94 677.50
4	2	$750.00	$426.05	$323.95	$94 353.55
5	3	$750.00	$424.59	$325.41	$94 028.14
6	4	$750.00	$423.13	$326.87	$93 701.27
7	5	$750.00	$421.66	$328.34	$93 372.92
188	184	$750.00	$16.55	$733.45	$2 944.92
187	185	$750.00	$13.25	$736.75	$2 208.17
188	186	$750.00	$9.94	$740.06	$1 468.11
189	187	$750.00	$6.61	$743.39	$724.72
190	188	$750.00	$3.26	$746.74	–$22.02

I used the FILL DOWN command to complete the spreadsheet until the balance was close to zero.

$$t = \frac{188}{12}$$

$$\doteq 15.667$$

After 188 payments, the balance is close to zero. I calculated the number of years needed to make 188 payments by dividing by 12, since there are 12 payments each year.

$$0.667 \times 12 \text{ months} \doteq 8 \text{ months}$$

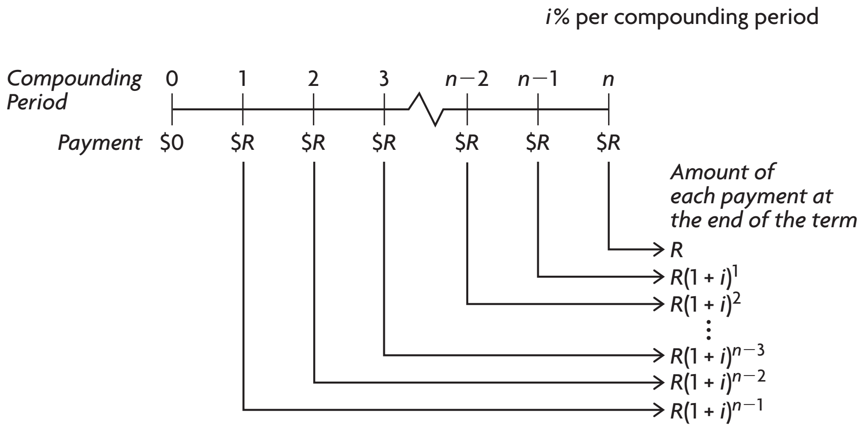
I got a value greater than 15. The 15 meant 15 years, so I had to figure out what 0.667 of a year was.

Nahid can pay off the loan after 188 payments, which would take about 15 years and 8 months.

In Summary

Key Ideas

- The future value of an annuity is the sum of all regular payments and interest earned.



- The future value can be written as the geometric series

$$FV = R + R(1 + i) + R(1 + i)^2 + \dots + R(1 + i)^{n-2} + R(1 + i)^{n-1}$$

  where FV is the future value; R is the regular payment; i is the interest rate per compounding period, expressed as a decimal; and n is the number of compounding periods.
- The formula for the sum of a geometric series can be used to determine the future value of an annuity.

Need to Know

- A variety of technological tools (spreadsheets, graphing calculators) can be used to solve problems involving annuities.
- The payment interval of an annuity is the time between successive payments.
- The term of an annuity is the time from the first payment to the last payment.
- The formula for the future value of an annuity is

$$FV = R \times \left(\frac{(1 + i)^n - 1}{i}\right)$$

  where FV is the future value; R is the regular payment each compounding period; i is the interest rate per compounding period, expressed as a decimal; and n is the number of compounding periods.

CHECK *Your Understanding*

1. Each year, Eric invests $2500 at 8.2%/a compounded annually for 25 years.
   a) Calculate the value of each of the first four investments at the end of 25 years.
   b) What type of sequence do the values form?
   c) Determine the total amount of all of Eric's investments.

2. Calculate the future value of each annuity.

	Regular Payment	Rate of Compound Interest per Year	Compounding Period	Time
a)	$100 per month	3.6%	monthly	50 years
b)	$1500 per quarter	6.2%	quarterly	15 years
c)	$500 every 6 months	5.6%	semi-annually	8 years
d)	$4000 per year	4.5%	annually	10 years

3. Lois invests $650 every 6 months at 4.6%/a compounded semi-annually for 25 years. How much interest will she earn after the 25th year?

4. Josh borrows some money on which he makes monthly payments of $125.43 for 3 years. If the interest rate is 5.4%/a compounded monthly, what will be the total amount of all of the payments at the end of the 3 years?

PRACTISING

5. Calculate the future value of each annuity.
K

	Regular Payment	Rate of Compound Interest per Year	Compounding Period	Time
a)	$1500 per year	6.3%	annually	10 years
b)	$250 every 6 months	3.6%	semi-annually	3 years
c)	$2400 per quarter	4.8%	quarterly	7 years
d)	$25 per month	8%	monthly	35 years

6. Mike wants to invest money every month for 40 years. He would like to have $1 000 000 at the end of the 40 years. For each investment option, how much does he need to invest each month?
A
   a) 10.2%/a compounded monthly
   b) 5.1%/a compounded monthly

7. Kiki has several options for investing \$1200 per year:

	Regular Payment	Rate of Compound Interest per Year	Compounding Period
a)	\$100 per month	7.2%	monthly
b)	\$300 per quarter	7.2%	quarterly
c)	\$600 every 6 months	7.2%	semi-annually
d)	\$1200 per year	7.2%	annually

Without doing any calculations, which investment would be best? Justify your reasoning.

8. Kenny wants to invest \$250 every three months at 5.2%/a compounded quarterly. He would like to have at least \$6500 at the end of his investment. How long will he need to make regular payments?

9. Sonja and Anita want to make equal monthly payments for the next 35 years. At the end of that time, each person would like to have \$500 000. Sonja's bank will give her 6.6%/a compounded monthly. Anita can invest through her work and earn 10.8%/a compounded monthly.

a) How much more per month does Sonja have to invest?

b) If Anita decides to invest the same monthly amount as Sonja, how much more money will she have at the end of 35 years?

10. **T** Jamal wants to invest \$150 every month for 10 years. At the end of that time, he would like to have \$25 000. At what annual interest rate, compounded monthly, does Jamal need to invest to reach his goal? Round your answer to two decimal places.

11. **C** Draw a mind map for the concept of *future value of annuities*. Show how it is related to interest, sequences, and series.

Extending

12. Carmen borrows \$10 000 at 4.8%/a compounded monthly. She decides to make monthly payments of \$250.

a) How long will it take her to pay off the loan?

b) How much interest will she pay over the term of the loan?

13. Greg borrows \$123 000 for the purchase of a house. He plans to make regular monthly payments over the next 20 years to pay off the loan. The bank is charging Greg 6.6%/a compounded monthly. What monthly payments will Greg have to make?

14. How many equal monthly payments would you have to make to get 100 times the amount you are investing each month if you are earning 8.4%/a compounded monthly?