8.3 Compound Interest: Present Value

GOAL

Determine the present value of an amount being charged or earning compound interest.

YOU WILL NEED

- graphing calculator
- spreadsheet software

LEARN ABOUT the Math

Anton's parents would like to put some money away so that he will have \$15 000 to study music professionally in 10 years. They can earn 6%/a compounded annually on their investment.

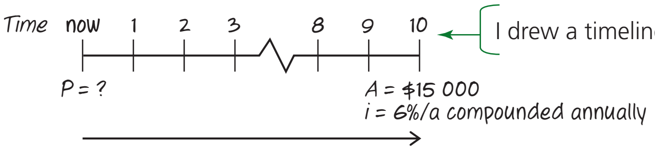
? How much money should Anton's parents invest now so that it will grow to \$15 000 in 10 years at 6%/a compounded annually?

EXAMPLE 1 Selecting a strategy to determine the principal for a given amount

Determine the **present value** of Anton's parents' investment if it must be worth \$15 000 ten years from now.

present value
the principal that would have to be invested now to get a specific future value in a certain amount of time. *PV* is used for present value instead of *P*, since *P* is used for principal.

Tina's Solution: Working Backward



I drew a timeline for the situation.

end of 1st year:

$I = 0.06P$

$A = P + 0.06P$

$= 1.06P$

The interest rate is 6%/a compounded annually. I calculated the interest earned and the amount at the end of the first year.

end of 2nd year:

$I = 0.06(1.06P)$

$A = 1.06P + 0.06(1.06P)$

$= 1.06P(1 + 0.06)$

$= 1.06P(1.06)$

$= 1.06^2P$

For the second year, interest is earned on the amount at the end of the first year. I calculated the interest earned and the amount at the end of the second year.

$1.06P, 1.06^2P, 1.06^3P, \ldots, 1.06^{10}P$

Since 6% is added at the end of each year, I got 106% of what I started with. So multiplying by 1.06 gives the next term of the sequence. The amounts at the end of each year form a geometric sequence with common ratio 1.06. The amount at the end of the 10th year has an exponent of 10.

end of 10th year:

$$15\ 000 = 1.06^{10}P$$

$$P = \frac{15\ 000}{1.06^{10}}$$

$$\doteq \$8375.92$$

Since Anton's parents want \$15 000 at the end of 10 years, I set the 10th term of the sequence equal to \$15 000 and solved for *P*.

Anton's parents would need to invest \$8375.92 now to get \$15 000 in 10 years.

Jamie's Solution: Representing the Formula for the Amount in a Different Way

$A = \$15\ 000$

$i = 6\% = 0.06$

$n = 10$

Anton's parents invest a certain amount and let it grow to \$15 000 at 6%/a compounded annually.

$A = PV(1 + i)^n$

An investment earning compound interest grows like an exponential function, so I wrote the amount as an exponential formula.

$$PV = \frac{A}{(1 + i)^n}$$

$$= \frac{15\ 000}{(1 + 0.06)^{10}}$$

$$\doteq \$8375.92$$

To calculate the principal that Anton's parents would have to invest, I rearranged the formula, substituted, and solved for *PV*.

Anton's parents would need to invest \$8375.92 now to get \$15 000 in 10 years.

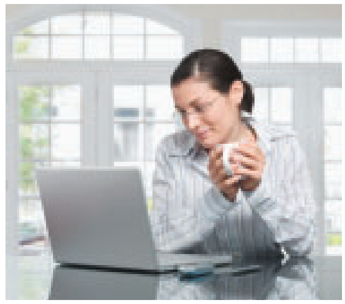
Reflecting

A. How are the problems of determining the present value and the amount of an investment related?

B. Based on Example 1, which method do you prefer to use to calculate the present value? Why?

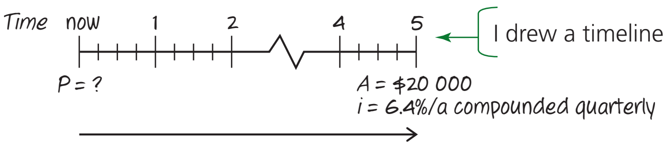
APPLY the Math

EXAMPLE 2 Solving a problem involving present value



Monica wants to start a business and needs to borrow some money. Her bank will charge her 6.4%/a compounded quarterly. Monica wants to repay the loan in 5 years, but doesn't want the amount she pays back to be more than $20 000. What is the maximum amount that she can borrow and how much interest will she pay if she doesn't pay anything back until the end of the 5 years?

Kwok's Solution



I drew a timeline for the situation.

$$i = \frac{0.064}{4} = 0.016$$

$$n = 5 \times 4 = 20$$

I calculated the interest rate Monica would be charged each compounding period and how many periods the loan would last.

$$PV = \frac{A}{(1 + i)^n}$$

$$= \frac{20\ 000}{(1 + 0.016)^{20}}$$

$$\doteq \$14\ 559.81$$

Next I calculated the present value of the $20 000 at the given interest rate.

$$I = A - PV$$

$$= \$20\ 000 - \$14\ 559.81$$

$$= \$5440.19$$

I determined the interest by subtracting the present value from the amount.

The most Monica can borrow is $14 559.81; she will pay $5440.19 in interest.

EXAMPLE 3 Selecting a strategy to determine the interest rate

Tony is investing \$5000 that he would like to grow to at least \$50 000 by the time he retires in 40 years. What annual interest rate, compounded annually, will provide this? Round your answer to two decimal places.

Philip's Solution: Using a Graphing Calculator

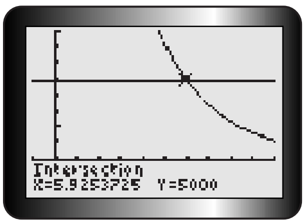
$$PV = \$5000$$
$$n = 40$$
$$A = 50\ 000$$

I knew the principal, the amount (or future value), and the number of years.

$$PV = \frac{A}{(1 + i)^n}$$
$$5000 = \frac{50\ 000}{(1 + i)^{40}}$$

I wrote the formula for the present value and substituted the given information. To calculate i, I thought of the intersection of two functions: $Y1 = \frac{50\ 000}{\left(1 + \frac{X}{100}\right)^{40}}$ and $Y2 = 5000$.

I entered these into my graphing calculator. I used Y1 and Y2 for the present value and X for the interest rate.



I graphed both functions on the same graph and used the calculator to find the point of intersection.

Tony would need to get at least 5.93%/a compounded annually to reach his goal.

Tech Support

For help using a graphing calculator to find the point of intersection of two functions, see Technical Appendix, B-12.

Derek's Solution: Using the Formula

$$PV = \$5000$$
$$n = 40$$
$$A = \$50\ 000$$

I knew the principal, the amount (or future value), and the number of years.

$$PV = \frac{A}{(1 + i)^n}$$
$$5000 = \frac{50\ 000}{(1 + i)^{40}}$$

I wrote the formula for the present value, then substituted the given information, and rearranged to solve for *i*.

$$5000(1 + i)^{40} = 50\ 000$$
$$(1 + i)^{40} = 10$$

I multiplied both sides of the equation by $(1 + i)^{40}$, then I divided both sides by 5000.

$$1 + i = \sqrt[40]{10}$$
$$1 + i \doteq 1.0593$$
$$i = 0.0593$$
$$i = 5.93\%$$

To calculate *i*, I used the inverse operation of raising something to the 40th power, which is determining the 40th root.

Tony would need to get at least 5.93%/a compounded annually to reach his goal.

In Summary

Key Idea

- The principal, *PV,* that must be invested now to grow to a specific future value, *A,* is called the present value.

Need to Know

- The present value of an investment earning compound interest can be calculated using the formula $PV = \frac{A}{(1 + i)^n}$ or $PV = A(1 + i)^{-n}$, where *PV* is the present value; *A* is the total amount, or future value; *i* is the interest rate per compounding period, expressed as a decimal; and *n* is the number of compounding periods.

CHECK Your Understanding

1. Calculate the present value of each investment.

	Rate of Compound Interest per Year	Compounding Period	Time	Future Value
a)	4%	annually	10 years	$10 000
b)	6.2%	semi-annually	5 years	$100 000
c)	5.2%	quarterly	15 years	$23 000
d)	6.6%	monthly	100 years	$2 500

2. Kevin and Lui both want to have $10 000 in 20 years. Kevin can invest at 5%/a compounded annually and Lui can invest at 4.8%/a compounded monthly. Who has to invest more money to reach his goal?

PRACTISING

3. For each investment, determine the present value and the interest earned.
K

	Rate of Compound Interest per Year	Compounding Period	Time	Future Value
a)	6%	annually	4 years	$10 000
b)	8.2%	semi-annually	3 years	$6 200
c)	5.6%	quarterly	15 years	$20 000
d)	4.2%	monthly	9 years	$12 800

4. Chandra borrows some money at 7.2%/a compounded annually. After 5 years, she repays $12 033.52 for the principal and the interest. How much money did Chandra borrow?

5. Nazir saved $900 to buy a plasma TV. He borrowed the rest at an interest rate of 18%/a compounded monthly. Two years later, he paid $1429.50 for the principal and the interest. How much did the TV originally cost?

6. Rico can invest money at 10%/a compounded quarterly. He would like $15 000 in 10 years. How much does he need to invest now?

7. Colin borrowed some money at 7.16%/a compounded quarterly. Three years later, he paid $5000 toward the principal and the interest. After another two years, he paid another $5000. After another five years, he paid the remainder of the principal and the interest, which totalled $5000. How much money did he originally borrow?

8. Tia is investing $2500 that she would like to grow to $6000 in 10 years. At what annual interest rate, compounded quarterly, must Tia invest her money? Round your answer to two decimal places.

9. **A** Franco invests some money at 6.9%/a compounded annually and David invests some money at 6.9%/a compounded monthly. After 30 years, each investment is worth $25 000. Who made the greater original investment and by how much?

10. **T** Sally invests some money at 6%/a compounded annually. After 5 years, she takes the principal and interest and reinvests it all at 7.2%/a compounded quarterly for 6 more years. At the end of this time, her investment is worth $14 784.56. How much did Sally originally invest?

11. Steve wants to have $25 000 in 25 years. He can get only 3.2%/a interest compounded quarterly. His bank will guarantee the rate for either 5 years or 8 years.

- In 5 years, he will probably get 4%/a compounded quarterly for the remainder of the term.
- In 8 years, he will probably get 5%/a compounded quarterly for the remainder of the term.

a) Which guarantee should Steve choose, the 5-year one or the 8-year one?
b) How much does he need to invest?

12. **C** Describe how determining the present value of an investment is similar to solving a radioactive decay problem.

Extending

13. Louise invests $5000 at 5.4%/a compounded semi-annually. She would like the money to grow to $12 000. How long will she have to wait?

14. What annual interest rate, compounded quarterly, would cause an investment to triple every 10 years? Round your answer to two decimal places.

15. You buy a home entertainment system on credit. You make monthly payments of $268.17 for $2\frac{1}{2}$ years and are charged 19.2%/a interest compounded monthly. How much interest will you have paid on your purchase?

16. Determine a formula for the present value of an investment with future value, A, earning simple interest at a rate of i% per interest period for n interest periods.

Curious | *Math*

The Rule of 72

Working with compound interest can be difficult if you don't have a calculator handy. For years, banks, investors, and the general public have used "the rule of 72" to help approximate calculations involving compound interest. Here is how the rule works:

If an investment is earning r%/a compounded annually, then it will take $72 \div r$ years to double in value.

So if you are earning 8%/a compounded annually, the rule indicates that it will take $72 \div 8 = 9$ years for the money to double in value. Suppose you invested \$1000. Then the formula for the future value gives

$$\begin{aligned} A &= P(1 + i)^n \\ &= 1000(1.08)^9 \\ &\doteq \$1999.00 \end{aligned}$$

which is very close to double your money. The spreadsheet below shows how the time, in years, predicted by the rule of 72 is very close to the actual doubling time.

	A	B	C
1	Interest Rate	Double Time Using the Rule of 72	Actual Double Time
2	1	72.00	69.66
3	2	36.00	35.00
4	3	24.00	23.45
5	4	18.00	17.67
6	5	14.40	14.21
7	6	12.00	11.90
8	7	10.29	10.24
9	8	9.00	9.01
10	9	8.00	8.04
11	10	7.20	7.27
12	11	6.55	6.64
13	12	6.00	6.12

1. How could you use the rule of 72 to determine how much a \$1000 investment earning 8%/a compounded annually would be worth after 45 years?
2. How close is your estimate to the actual amount after 45 years?