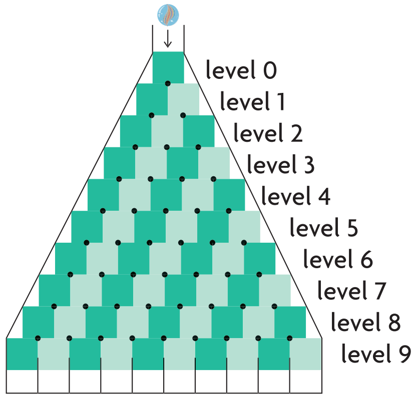
7.7 Pascal's Triangle and Binomial Expansions

GOAL

Investigate patterns in Pascal's triangle, and use one of these patterns to expand binomials efficiently.

INVESTIGATE the Math

A child's toy called "Rockin' Rollers" involves dropping a marble into its top. When the marble hits a pin, it has the same chance of going either left or right. A version of the toy with nine levels is shown at the right.



? How many paths are there to each of the bins at the bottom of this version of "Rockin' Rollers"?

A. Consider a "Rockin' Rollers" toy that has only one level. Calculate the number of paths to each bin at the bottom. Repeat the calculation with a toy having two and three levels.

B. How is the number of paths for a toy with three levels related to the number of paths for a toy with two levels? Why is this so?

C. Use the pattern to predict how many paths lead to each bin in a toy with four levels. Check your prediction by counting the number of paths.

D. Use your pattern to calculate the number of paths to each bin in a toy with nine levels.

Reflecting

E. How is the number of paths for each bin in a given level related to the number of paths in the level above it?

F. The triangular pattern of numbers in the "Rockin' Rollers" toy is known as Pascal's triangle, named after French mathematician Blaise Pascal (1623–62), who explored many of its properties. What other pattern(s) can you find in Pascal's triangle?

Blaise Pascal

APPLY the Math

EXAMPLE 1 Connecting Pascal's triangle to the expansion of a binomial power

Expand $(x + y)^6$.

Pedro's Solution

$$(x + y)^1 = 1x + 1y$$

The binomial to the 1st power is the same as the binomial itself.

$$(x + y)^2 = 1x^2 + 2xy + 1y^2$$

I expanded the square of a binomial.

$$\begin{aligned}(x + y)^3 &= (x + y)(x + y)^2 \\ &= (x + y)(x^2 + 2xy + y^2) \\ &= x^3 + 2x^2y + xy^2 + x^2y + 2xy^2 + y^3 \\ &= 1x^3 + 3x^2y + 3xy^2 + 1y^3\end{aligned}$$

I expanded the binomial to the 3rd power. I noticed that the coefficients in each of the expansions so far are numbers in each row of Pascal's triangle.

1
1 1
1 2 1
1 3 3 1

Each term in the expansion is in terms of a product of an exponent of x and an exponent of y. The exponents of x start from 3 (the exponent of the binomial), and go down to zero, while the exponents of y start at zero and go up to 3. In each term, the sum of the x and y exponents is always 3.

1	0th row
1 1	1st row
1 2 1	2nd row
1 3 3 1	3rd row
1 4 6 4 1	4th row
1 5 10 10 5 1	5th row

I wrote out 6 rows of Pascal's triangle. The one at the top must correspond to $(x + y)^0 = 1$.

$$\begin{aligned}(x + y)^4 &= (x + y)(x + y)^3 \\ &= (x + y)(x^3 + 3x^2y + 3xy^2 + y^3) \\ &= x^4 + 3x^3y + 3x^2y^2 + xy^3 + 3x^2y^2 + 3xy^3 + y^4 \\ &= x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4\end{aligned}$$

I tried one more expansion to check that these patterns continue.

$$(x + y)^6 = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^3 + 6xy^5 + y^6$$

To expand $(x + y)^6$, I used my patterns and the 6th row of the triangle.

Any binomial can be expanded by using Pascal's triangle to help determine the coefficients of each term.

EXAMPLE 2 Selecting a strategy to expand a binomial power involving a variable in one term

Expand and simplify $(x - 2)^5$.

Tanya's Solution

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

Since the exponent of the binomial is 5, I used the 5th row of Pascal's triangle to determine the coefficients.

$$(x-2)^5 = 1(x)^5 + 5(x)^4(-2)^1 + 10(x)^3(-2)^2 + 10(x)^2(-2)^3 + 5(x)^1(-2)^4 + 1(-2)^5$$

I used the terms x and -2 and applied the pattern for expanding a binomial. The exponents in each term always add up to 5. As the x exponents decrease by 1 each time, the exponents of -2 increase by 1.

$$= x^5 + 5(x^4)(-2) + 10(x^3)(4) + 10(x^2)(-8) + 5(x)(16) + (-32)$$

I simplified each term.

$$= x^5 - 10x^4 + 40x^3 - 80x^2 + 80x - 32$$

EXAMPLE 3 Selecting a strategy to expand a binomial power involving a variable in each term

Expand and simplify $(5x + 2y)^3$.

Jason's Solution

1
1 1
1 2 1
1 3 3 1

Since the exponent of the binomial is 3, I wrote out the 3rd row of Pascal's triangle.

$$(5x + 2y)^3$$
$$= 1(5x)^3 + 3(5x)^2(2y)^1 + 3(5x)^1(2y)^2 + 1(2y)^3$$

I used the terms $5x$ and $2y$ and applied the pattern for expanding a binomial.

$$= 1(125x^3) + 3(25x^2)(2y) + 3(5x)(4y^2) + 1(8y^3)$$

I simplified each term.

$$= 125x^3 + 150x^2y + 60xy^2 + 8y^3$$

In Summary

Key Ideas

- The arrangement of numbers shown is called Pascal's triangle. Each row is generated by calculating the sum of pairs of consecutive terms in the previous row.

1	0th row
1 1	1st row
1 2 1	2nd row
1 3 3 1	3rd row
1 4 6 4 1	4th row
1 5 10 10 5 1	5th row

- The numbers in Pascal's triangle correspond to the coefficients in the expansion of binomials raised to whole-number exponents.

Need to Know

- Pascal's triangle has many interesting relationships among its numbers. Some of these relationships are recursive.
  - For example, down the sides are constant sequences: 1, 1, 1, ...
  - The diagonal beside that is the counting numbers, 1, 2, 3, ... , which form an arithmetic sequence.
  - The next diagonal is the triangular numbers, 1, 3, 6, 10, ... , which can be defined by the recursive formula

$$t_1 = 1, t_n = t_{n-1} + n, \text{ where } n \in \mathbf{N} \text{ and } n > 1.$$

- There are patterns in the expansions of a binomial $(a + b)^n$:
  - Each term in the expansion is the product of a number from Pascal's triangle, a power of a, and a power of b.
  - The coefficients in the expansion correspond to the numbers in the nth row in Pascal's triangle.
  - In the expansion, the exponents of a start at n and decrease by 1 down to zero, while the exponents of b start at zero and increase by 1 up to n.
  - In each term, the sum of the exponents of a and b is always n.

CHECK Your Understanding

1. The first four entries of the 12th row of Pascal's triangle are 1, 12, 66, and 220. Determine the first four entries of the 13th row of the triangle.

2. Expand and simplify each binomial power.
   a) $(x + 2)^5$ b) $(x - 1)^6$ c) $(2x - 3)^3$

3. Expand and simplify the first three terms of each binomial power.
   a) $(x + 5)^{10}$ b) $(x - 2)^8$ c) $(2x - 7)^9$

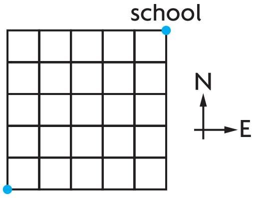
PRACTISING

4. Expand and simplify each binomial power.
   K a) $(k + 3)^4$ c) $(3q - 4)^4$ e) $\left(\sqrt{2}x + \sqrt{3}\right)^6$
   b) $(y - 5)^6$ d) $(2x + 7y)^3$ f) $(2z^3 - 3y^2)^5$

5. Expand and simplify the first three terms of each binomial power.
   a) $(x - 2)^{13}$ c) $(z^5 - z^3)^{11}$ e) $\left(3b^2 - \frac{2}{b}\right)^{14}$
   b) $(3y + 5)^9$ d) $\left(\sqrt{a} + \sqrt{5}\right)^{10}$ f) $(5x^3 + 3y^2)^8$

6. Using the pattern for expanding a binomial, expand each binomial power to describe a pattern in Pascal's triangle.
   a) $2^n = (1 + 1)^n$
   b) $0 = (1 - 1)^n$, where $n \geq 1$

7. Using the pattern for expanding a binomial, expand and simplify the expression $\frac{1}{\sqrt{5}}\left[\left(\frac{1 + \sqrt{5}}{2}\right)^n - \left(\frac{1 - \sqrt{5}}{2}\right)^n\right]$, where $n = 1, 2, 3$, and 4. How are the terms related?



8. **A** Using the diagram at the left, determine the number of different ways that Joan could walk to school from her house if she always travels either north or east.

9. **T** Explain, without calculating, how you can use the pattern for expanding a binomial to expand $(x + y + z)^{10}$.

10. Expand and simplify $(3x - 5y)^6$.

11. **C** Summarize the methods of expanding a binomial power and determining a term in an expansion.

Extending

12. If a relation is linear, the 1st differences are constant. If the 2nd differences are also constant, the relation is quadratic. Use the pattern for expanding a binomial to demonstrate that if a relation is cubic, the third differences are constant. (*Hint:* You may want to look at x^3 and $(x + 1)^3$.)

13. When a fair coin is tossed, the probability of getting heads or tails is $\frac{1}{2}$. Expand and simplify the first three terms in the expression $\left(\frac{1}{2} + \frac{1}{2}\right)^{10}$. How are the terms related to tossing the coin 10 times?