7.6 Geometric Series

YOU WILL NEED

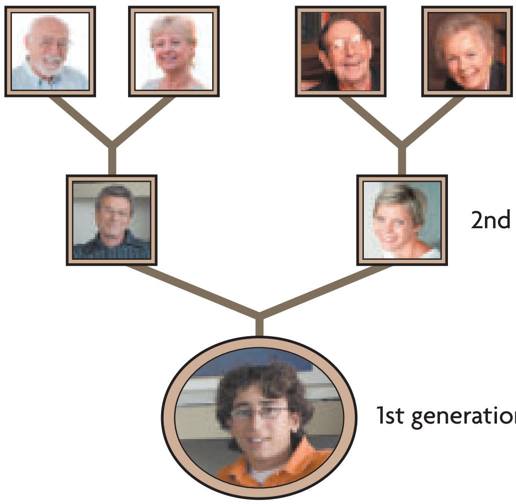
- spreadsheet software

GOAL

Calculate the sum of the terms of a geometric sequence.

INVESTIGATE the Math

An ancestor tree is a family tree that shows only the parents in each generation. John started to draw his ancestor tree, starting with his own parents. His complete ancestor tree includes 13 generations.



? How many people are in John's ancestor tree?

A. Create a sequence to represent the number of people in each generation for the first six generations. How do you know that this sequence is geometric?

B. Based on your sequence, create a geometric series to represent the total number of people in John's ancestor tree.

geometric series
the sum of the terms of a geometric sequence

C. Write the series again, but this time multiply each term by the common ratio. Write both series, rS_n and S_n, so that equal terms are aligned one above the other. Subtract S_n from rS_n.

D. Based on your calculation in part C, determine how many people are in John's ancestor tree.

Reflecting

E. How is the sum of a geometric series related to an exponential function?

F. Why did lining up equal terms make the subtraction easier?

APPLY the Math

EXAMPLE 1 | Representing the sum of a geometric series

Determine the sum of the first n terms of the geometric series.

Tara's Solution

$$t_n = ar^{n-1}$$

The series is geometric. To find S_n, I added all terms up to t_n. The nth term of the series corresponds to the general term of a geometric sequence.

$$\begin{aligned} rS_n &= \quad\;\; ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1} + ar^n \\ -S_n &= -(a + ar + ar^2 + ar^3 + \ldots + ar^{n-2} + ar^{n-1}) \\ \hline (r-1)S_n &= -a + 0 + 0 + 0 + \ldots + 0 + 0 + ar^n \\ (r-1)S_n &= -a + ar^n \end{aligned}$$

I wrote the sum out. If I multiplied every term by the common ratio, most of the terms would be repeated. I wrote this new series above the original series and lined up equal terms, so I would get zero for most of the terms when I subtracted. Only the first term of S_n and the last term of rS_n would remain.

$$(r-1)S_n = a(r^n - 1)$$

$$\left(\frac{r-1}{r-1}\right)S_n = \frac{a(r^n-1)}{r-1}$$

$$S_n = \frac{a(r^n-1)}{r-1}$$

I solved for S_n by dividing both sides by $r - 1$.

The sum of the first n terms of a geometric series is

$$S_n = \frac{a(r^n-1)}{r-1}, \text{ where } r \neq 1.$$

$$S_n = \frac{ar^n - a}{r-1}$$

$$S_n = \frac{t_{n+1} - t_1}{r-1}, \text{ where } r \neq 1.$$

I wrote this formula another way by expanding the numerator. I noticed that a is the first term in the series and ar^n is the $(n + 1)$th term.

If a problem involves adding together the terms of a geometric sequence, you can use the formula for the sum of geometric series.

EXAMPLE 2 Solving a problem by using a geometric series

At a fish hatchery, fish hatch at different times even though the eggs were all fertilized at the same time. The number of fish that hatched on each of the first four days after fertilization was 2, 10, 50, and 250, respectively. If the pattern continues, calculate the total number of fish hatched during the first 10 days.

Joel's Solution

$$\frac{t_2}{t_1} = \frac{10}{2} = 5 \qquad \frac{t_3}{t_2} = \frac{50}{10} = 5 \qquad \frac{t_4}{t_3} = \frac{250}{50} = 5$$

$$\therefore r = 5$$

I checked to see if the sequence 2, 10, 50, 250, ... is geometric. So I calculated the ratio of consecutive terms. Since all the ratios are the same, the sequence is geometric.

$$a = 2$$

$$n = 10$$

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

$$S_{10} = \frac{2(5^{10} - 1)}{5 - 1}$$

$$= 4\ 882\ 812$$

The first term is 2 and there are 10 terms. Since I knew the first term, the common ratio, and the number of terms, I used the formula for the sum of a geometric series in terms of a, r, and n. I substituted $a = 2$, $r = 5$, and $n = 10$.

A total of 4 882 812 fish hatched during the first 10 days.

EXAMPLE 3 Selecting a strategy to calculate the sum of a geometric series when the number of terms is unknown

Calculate the sum of the geometric series
7 971 615 + 5 314 410 + 3 542 940 + ... + 92 160.

Jasmine's Solution: Using a Spreadsheet

$$\frac{t_2}{t_1} = \frac{5\,314\,410}{7\,971\,615}$$

$$= \frac{2}{3}$$

$$\therefore r = \frac{2}{3}$$

I knew that the series is geometric. So I calculated the common ratio.

	A	B
1	n	tn
2	1	7971615
3	2	5314410
4	3	3542940
5	4	2361960
6	5	1574640
7	6	1049760
8	7	699840
9	8	466560
10	9	311040
11	10	207360
12	11	138240
13	12	92160
14	13	61440

I needed to determine the number of terms, n, to get to $t_n = 92\,160$. So I set up a spreadsheet to generate the terms of the series. I saw that the 12th term is 92 160.

$$S_n = \frac{t_{n+1} - t_1}{r - 1}$$

$$S_{12} = \frac{61\,440 - 7\,971\,615}{\frac{2}{3} - 1}$$

$$= 23\,730\,525$$

From the spreadsheet, 61 440 corresponds to the $(n + 1)$th term. Since I knew the first term and the $(n + 1)$th term, I used the formula for the sum of a geometric series in terms of t_1 and t_{n+1}.

The sum of the series 7 971 615 + 5 314 410 + 3 542 940 + ... + 92 160 is 23 730 525.

Mario's Solution: Using a Graphing Calculator

$$\frac{t_2}{t_1} = \frac{5\ 314\ 410}{7\ 971\ 615}$$

$$= \frac{2}{3}$$

$$\therefore r = \frac{2}{3}$$

I knew that the series is geometric. So I calculated the common ratio.

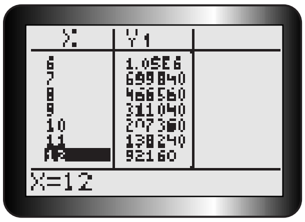
$$t_n = ar^{n-1}$$

$$= 7\ 971\ 615 \times \left(\frac{2}{3}\right)^{n-1}$$

I wrote the formula for the general term of a geometric sequence. I substituted $a = 7\ 971\ 615$ and $r = \frac{2}{3}$ into the formula.

$$92\ 160 = 7\ 971\ 615 \times \left(\frac{2}{3}\right)^{n-1}$$

I needed to determine the number of terms, n, to get to $t_n = 92\ 160$.



I entered the function Y1 = 7 971 615(2/3)^(X−1) into my graphing calculator. Then I used the table function to determine the term number whose value was 92 160. It was $n = 12$.

$$S_n = \frac{a(r^n - 1)}{r - 1}$$

Since I knew that $a = 7\ 971\ 615$, $r = \frac{2}{3}$, and $n = 12$, I substituted these values into the formula for the sum of a geometric series in terms of a, r, and n. I used my calculator to evaluate.

The sum of the series 7 971 615 + 5 314 410 + 3 542 940 + ... + 92 160 is 23 730 525.

Tech Support

For help using the table on a graphing calculator, see Technical Appendix, B-6.

In Summary

Key Idea

- A geometric series is created by adding the terms of a geometric sequence. For the sequence 3, 6, 12, 24, ... , the related geometric series is $3 + 6 + 12 + 24 + ...$

Need to Know

- The sum of the first n terms of a geometric sequence can be calculated using
  - $S_n = \dfrac{a(r^n - 1)}{r - 1}$, where $r \neq 1$ or
  - $S_n = \dfrac{t_{n+1} - t_1}{r - 1}$, where $r \neq 1$.

  In both cases, $n \in \mathbf{N}$, a is the first term, and r is the common ratio.
- You can use either formula, but you need to know the common ratio and the first term. If you know the $(n + 1)$th term, use the formula in terms of t_1 and t_{n+1}. If you can calculate the number of terms, use the formula in terms of a, r, and n.

CHECK Your Understanding

1. Calculate the sum of the first seven terms of each geometric series.

a) $6 + 18 + 54 + ...$

b) $100 + 50 + 25 + ...$

c) $8 - 24 + 72 - ...$

d) $\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + ...$

2. Calculate the sum of the first six terms of a geometric sequence with first term 11 and common ratio 4.

PRACTISING

3. For each geometric series, calculate t_6 and S_6.

a) $6 + 30 + 150 + ...$

b) $-11 - 33 - 99 - ...$

c) $21\ 000\ 000 + 4\ 200\ 000 + 840\ 000 + ...$

d) $\frac{4}{5} + \frac{8}{15} + \frac{16}{45} + ...$

e) $3.4 - 7.14 + 14.994 - ...$

f) $1 + 3x^2 + 9x^4 + ...$

4. **K** **i)** Determine whether each series is arithmetic, geometric, or neither.

ii) If the series is geometric, calculate the sum of the first eight terms.

a) $5 + 10 + 15 + 20 + ...$

b) $7 + 21 + 63 + 189 + ...$

c) $2048 - 512 + 128 - 32 + ...$

d) $10 - 20 + 30 - 40 + ...$

e) $1.1 + 1.21 + 1.331 + 1.4641 + ...$

f) $81 + 63 + 45 + 27 + ...$

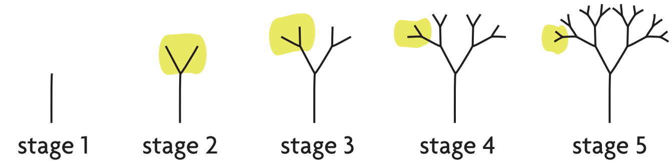
5. Determine the sum of the first seven terms of the geometric series in which
   a) $t_1 = 13$ and $r = 5$
   b) the first term is 11 and the seventh term is 704
   c) $t_1 = 120$ and $t_2 = 30$
   d) the third term is 18 and the terms increase by a factor of 3
   e) $t_8 = 1024$ and the terms decrease by a factor of $\frac{2}{3}$
   f) $t_5 = 5$ and $t_8 = -40$

6. Calculate the sum of each geometric series.
   a) $1 + 6 + 36 + \ldots + 279\,936$
   b) $960 + 480 + 240 + \ldots + 15$
   c) $17 - 51 + 153 - \ldots - 334\,611$
   d) $24\,000 + 3600 + 540 + \ldots + 1.8225$
   e) $-6 + 24 - 96 + \ldots + 98\,304$
   f) $4 + 2 + 1 + \ldots + \frac{1}{1024}$

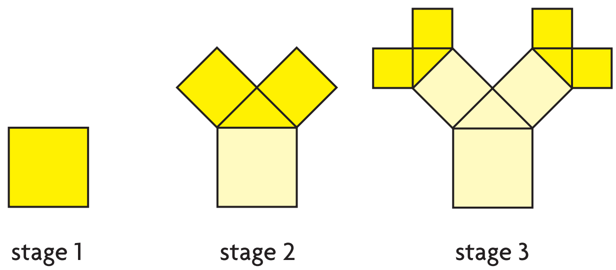
7. A ball is dropped from a height of 3 m and bounces on the ground. At the top of each bounce, the ball reaches 60% of its previous height. Calculate the total distance travelled by the ball when it hits the ground for the fifth time.

8. The formula for the sum of a geometric series is $S_n = \frac{a(r^n - 1)}{r - 1}$ or $S_n = \frac{t_{n+1} - t_1}{r - 1}$, each of which is valid only if $r \neq 1$. Explain how you would determine the sum of a geometric series if $r = 1$.

9. A A simple fractal tree grows in stages. At each new stage, two new line segments branch out from each segment at the top of the tree. The first five stages are shown. How many line segments need to be drawn to create stage 20?



10. A Pythagorean fractal tree starts at stage 1 with a square of side length 1 m. At every consecutive stage, an isosceles right triangle and two squares are attached to the last square(s) drawn. The first three stages are shown. Calculate the area of the tree at the 10th stage.



11. A large company has a phone tree to contact its employees in case of an emergency factory shutdown. Each of the five senior managers calls three employees, who each call three other employees, and so on. If the tree consists of seven levels, how many employees does the company have?

12. John wants to calculate the sum of a geometric series with 10 terms, where the 10th term is 5 859 375 and the common ratio is $\frac{5}{3}$. John solved the problem by considering another geometric series with common ratio $\frac{3}{5}$. Use John's method to calculate the sum. Justify your reasoning.

13. T A cereal company attempts to promote its product by placing certificates for a cash prize in selected boxes. The company wants to come up with a number of prizes that satisfy all of these conditions:

a) The total of the prizes is at most \$2 000 000.

b) Each prize is in whole dollars (no cents).

c) When the prizes are arranged from least to greatest, each prize is a constant integral multiple of the next smaller prize and is
- more than double the next smaller prize
- less than 10 times the next smaller prize

Determine a set of prizes that satisfies these conditions.

14. C Describe several methods for calculating the partial sums of an arithmetic and a geometric series. How are the methods similar? different?

Extending

15. In a geometric series, $t_1 = 12$ and $S_3 = 372$. What is the greatest possible value for t_5? Justify your answer.

16. In a geometric series, $t_1 = 23$, $t_3 = 92$, and the sum of all of the terms of the series is 62 813. How many terms are in the series?

17. Factor $x^{15} - 1$.

18. Suppose you want to calculate the sum of the *infinite* geometric series
$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots.$$

a) The diagram shown illustrates the first term of this series. Represent the next three terms on the diagram.

b) How can the formula for the sum of a geometric series be used in this case?

c) Does it make sense to talk about adding together an infinite number of terms? Justify your reasoning.