7.5 Arithmetic Series

YOU WILL NEED

- linking cubes

GOAL

Calculate the sum of the terms of an arithmetic sequence.

INVESTIGATE the Math

Marian goes to a party where there are 23 people present, including her. Each person shakes hands with every other person once and only once.

? How can Marian determine the total number of handshakes that take place?

A. Suppose the people join the party one at a time. When they enter, they shake hands with the host and everyone who is already there. Create a sequence representing the number of handshakes each person will make. What type of sequence is this?

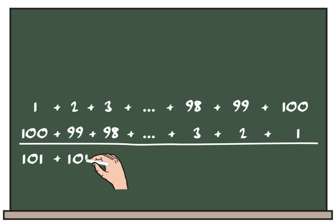
B. Write your sequence from part A, but include plus signs between terms. This expression is a **series** and represents the total number of handshakes.

C. When German mathematician Karl Friedrich Gauss (1777–1855) was a child, his teacher asked him to calculate the sum of the numbers from 1 to 100. Gauss wrote the list of numbers twice, once forward and once backward. He then paired terms from the two lists to solve the problem. Use this method to determine the sum of your **arithmetic series.**

D. Solve the handshake problem without using Gauss's method.

series
the sum of the terms of a sequence

arithmetic series
the sum of the terms of an arithmetic sequence



Reflecting

E. Suppose the partial sums of an arithmetic series are the terms of an arithmetic sequence. What would you notice about the 1st and 2nd differences?

F. Why is Gauss's method for determining the sum of an arithmetic series efficient?

G. Consider the arithmetic series $1 + 6 + 11 + 16 + 21 + 26 + 31 + 36$. Use Gauss's method to determine the sum of this series. Do you think this method will work for *any* arithmetic series? Justify your answer.

partial sum
the sum, S_n, of the first n terms of a sequence

APPLY the Math

EXAMPLE 1 Representing the sum of an arithmetic series

Determine the sum of the first n terms of the arithmetic series
$a + (a + d) + (a + 2d) + (a + 3d) + \ldots$.

Barbara's Solution

$$S_n = a + (a + d) + (a + 2d) + (a + 3d) + \ldots + t_n$$

The series is arithmetic. To find S_n, I added all terms up to t_n. The nth term of the series corresponds to the general term of an arithmetic sequence, $t_n = a + (n - 1)d$.

$$\begin{aligned} S_n &= a + (a + d) + \ldots + [a + (n - 2)d] + [a + (n - 1)d] \\ + S_n &= [a + (n - 1)d] + [a + (n - 2)d] + \ldots + (a + d) + a \\ \hline 2S_n &= [2a + (n - 1)d] + [2a + (n - 1)d] + \ldots + [2a + (n - 1)d] + [2a + (n - 1)d] \end{aligned}$$

$$2S_n = n \times [2a + (n - 1)d]$$

$$S_n = \frac{n[2a + (n - 1)d]}{2}$$

Using Gauss's method, I wrote the sum out twice, first forward and then backward. Next, I added each column. Since the terms in the top row go *up* by d and the terms in the bottom row go *down* by d, each pair of terms has the same sum.

There are n pairs that add up to $2a + (n - 1)d$, but that represents $2S_n$, so I divided by 2.

The sum of the first n terms of an arithmetic series is

$$\begin{aligned} S_n &= \frac{n[2a + (n - 1)d]}{2} \\ &= \frac{n[a + a + (n - 1)d]}{2} \\ &= \frac{n[a + (a + (n - 1)d)]}{2} \\ &= \frac{n(t_1 + t_n)}{2} \end{aligned}$$

I knew that $2a = a + a$, so I wrote this formula another way. I regrouped the terms in the numerator. I noticed that a is the first term of the series and $a + (n - 1)d$ is the nth term.

If a problem involves adding the terms of an arithmetic sequence, you can use the formula for the sum of an arithmetic series.

EXAMPLE 2 Solving a problem by using an arithmetic series

In an amphitheatre, seats are arranged in 50 semicircular rows facing a domed stage. The first row contains 23 seats, and each row contains 4 more seats than the previous row. How many seats are in the amphitheatre?

Kew's Solution

$a = 23, d = 4$ — Since each row has 4 more seats than the previous row, the number of seats in each row forms an arithmetic sequence.

$23 + 27 + 31 + ... + t_{50}$ — I wrote an arithmetic series to represent the total number of seats in the amphitheatre.

$$S_n = \frac{n[2a + (n - 1)d]}{2}$$

$$S_{50} = \frac{(50)[2(23) + (50 - 1)(4)]}{2}$$

$$= 6050$$

Since I knew the first term and the common difference, I used the formula for the sum of an arithmetic series in terms of *a* and *d*. I substituted $n = 50$ since there are 50 rows of seats.

There are 6050 seats in the amphitheatre.

In order to determine the sum of any arithmetic series, you need to know the number of terms in the series.

EXAMPLE 3 Selecting a strategy to calculate the sum of a series when the number of terms is unknown

Determine the sum of $-31 - 35 - 39 - ... -403$.

Jasmine's Solution

$$t_2 - t_1 = -35 - (-31) = -4$$

$$t_3 - t_2 = -39 - (-35) = -4$$

I checked to see if the series was arithmetic. So I calculated a few 1st differences. The differences were the same, so the series is arithmetic.

$$t_n = a + (n - 1)d$$
$$-403 = -31 + (n - 1)(-4)$$
$$-403 + 31 = (n - 1)(-4)$$
$$-372 = (n - 1)(-4)$$
$$\frac{-372}{-4} = \frac{(n - 1)(-4)}{-4}$$
$$93 = n - 1$$
$$93 + 1 = n$$
$$94 = n$$

I needed to determine the value of n when $t_n = -403$. So I substituted $a = 31$, $d = 4$, and t_n into the formula for the general term of an arithmetic sequence and solved for n.

There are 94 terms in this sequence.

$$S_n = \frac{n(t_1 + t_n)}{2}$$
$$S_{94} = \frac{94[-31 + (-403)]}{2}$$
$$= -20\,398$$

Since I knew the first and last terms of the series, I used the formula for the sum of an arithmetic series in terms of t_1 and t_n. I substituted $n = 94$, $t_1 = -31$, and $t_{94} = -403$.

The sum of the series $-31 - 35 - 39 - \ldots -403$ is $-20\,398$.

In Summary

Key Idea

- An arithmetic series is created by adding the terms of an arithmetic sequence. For the sequence 2, 10, 18, 26, ... , the related arithmetic series is 2 + 10 + 18 + 26 + ...
- The partial sum, S_n, of a series is the sum of a finite number of terms from the series, $S_n = t_1 + t_2 + t_3 + \ldots + t_n$.
  For example, for the sequence 2, 10, 18, 26, ... ,

$$S_4 = t_1 + t_2 + t_3 + t_4$$
$$= 2 + 10 + 18 + 26$$
$$= 56$$

Need to Know

- The sum of the first n terms of an arithmetic sequence can be calculated using
  - $S_n = \frac{n[2a + (n - 1)d]}{2}$ or
  - $S_n = \frac{n(t_1 + t_n)}{2}$.

  In both cases, $n \in \mathbf{N}$, a is the first term, and d is the common difference.
- You can use either formula, but you need to know the number of terms in the series and the first term. If you know the last term, use the formula in terms of t_1 and t_n. If you can calculate the common difference, use the formula in terms of a and d.

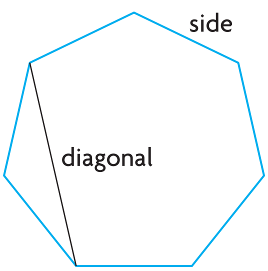
CHECK Your Understanding

1. Calculate the sum of the first 10 terms of each arithmetic series.
   a) $59 + 64 + 69 + \dots$
   b) $31 + 23 + 15 + \dots$
   c) $-103 - 110 - 117 - \dots$
   d) $-78 - 56 - 34 - \dots$

2. Calculate the sum of the first 20 terms of an arithmetic sequence with first term 18 and common difference 11.
3. Bricks are stacked in 20 rows such that each row has a fixed number of bricks more than the row above it. The top row has 5 bricks and the bottom row 62 bricks. How many bricks are in the stack?

PRACTISING

4. i) Determine whether each series is arithmetic.
   ii) If the series is arithmetic, calculate the sum of the first 25 terms.
   a) $-5 + 1 + 7 + 13 + \dots$
   b) $2 + 10 + 50 + 250 + \dots$
   c) $1 + 1 + 2 + 3 + \dots$
   d) $18 + 22 + 26 + 30 + \dots$
   e) $31 + 22 + 13 + 4 + \dots$
   f) $1 - 3 + 5 - 7 + \dots$
5. For each series, calculate t_{12} and S_{12}.
   K
   a) $37 + 41 + 45 + 49 + \dots$
   b) $-13 - 24 - 35 - 46 - \dots$
   c) $-18 - 12 - 6 + 0 + \dots$
   d) $\frac{1}{5} + \frac{7}{10} + \frac{6}{5} + \frac{17}{10} + \dots$
   e) $3.19 + 4.31 + 5.43 + 6.55 + \dots$
   f) $p + (2p + 2q) + (3p + 4q) + (4p + 6q) + \dots$
6. Determine the sum of the first 20 terms of the arithmetic series in which
   a) the first term is 8 and the common difference is 5
   b) $t_1 = 31$ and $t_{20} = 109$
   c) $t_1 = 53$ and $t_2 = 37$
   d) the 4th term is 18 and the terms increase by 17
   e) the 15th term is 107 and the terms decrease by 3
   f) the 7th term is 43 and the 13th term is 109
7. Calculate the sums of these arithmetic series.
   a) $1 + 6 + 11 + \dots + 96$
   b) $24 + 37 + 50 + \dots + 349$
   c) $85 + 77 + 69 + \dots - 99$
   d) $5 + 8 + 11 + \dots + 2135$
   e) $-31 - 38 - 45 - \dots - 136$
   f) $-63 - 57 - 51 - \dots + 63$



8. A diagonal in a regular polygon is a line segment joining two nonadjacent vertices.
   A
   a) Develop a formula for the number of diagonals for a regular polygon with n sides.
   b) Show that your formula works for a regular heptagon (a seven-sided polygon).

9. Joe invests \$1000 at the start of each year for five years and earns 6.3% simple interest on his investments. How much will all his investments be worth at the start of the fifth year?

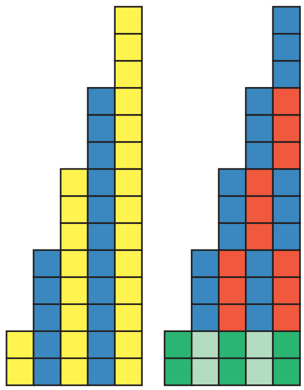
10. During a skydiving lesson, Chandra jumps out of a plane and falls 4.9 m during the first second. For each second afterward, she continues to fall 9.8 m more than the previous second. After 15 s, she opens her parachute. How far did Chandra fall before she opened her parachute?

11. Jamal got a job working on an assembly line in a toy factory. On the 20th day of work, he assembled 137 toys. He noticed that since he started, every day he assembled 3 more toys than the day before. How many toys did Jamal assemble altogether during his first 20 days?

12. In the video game "Geometric Constructors," a number of shapes have to be arranged into a predefined form. In level 1, you are given 3 min 20 s to complete the task. At each level afterward, a fixed number of seconds are removed from the time until, at level 20, 1 min 45 s are given. What would be the total amount of time given if you were to complete the first 20 levels?

13. **T** Sara is training to run a marathon. The first week she runs 5 km each day. The next week, she runs 7 km each day. During each successive week, each day she runs 2 km farther than she ran the days of the previous week. If she runs for five days each week, what total distance will Sara run in a 10 week training session?

14. **C** Joan is helping a friend understand the formulas for an arithmetic series. She uses linking cubes to represent the sum of the series $2 + 5 + 8 + 11 + 14$ two ways. These representations are shown at the right. Explain how the linking-cube representations can be used to explain the formulas for an arithmetic series.



Extending

15. The 10th term of an arithmetic series is 34, and the sum of the first 20 terms is 710. Determine the 25th term.

16. The arithmetic series $1 + 4 + 7 + \ldots + t_n$ has a sum of 1001. How many terms does the series have?