7.4 Exploring Recursive Sequences

GOAL

Explore patterns in sequences in which a term is related to the previous two terms.

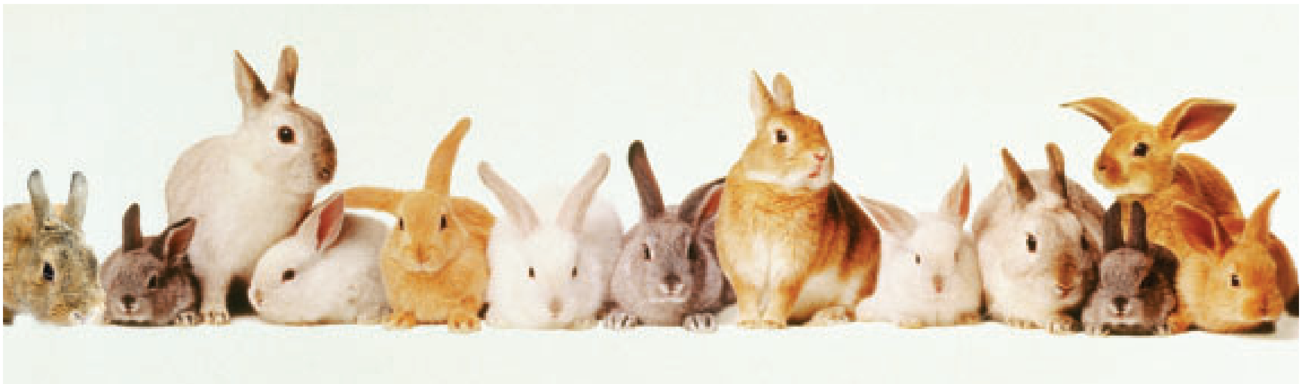
YOU WILL NEED

- graph paper

EXPLORE the Math

In his book *Liber Abaci* (*The Book of Calculation*), Italian mathematician Leonardo Pisano (1170–1250), nicknamed Fibonacci, described a situation like this:

A man put a pair of newborn rabbits (one male and one female) in an area surrounded on all sides by a wall. When the rabbits are in their second month of life, they produce a new pair of rabbits every month (one male and one female), which eventually mate. If the cycle continues, how many pairs of rabbits are there every month?



The sequence that represents the number of pairs of rabbits each month is called the Fibonacci sequence in Pisano's honour.

? What relationships can you determine in the Fibonacci sequence?

A. The first five terms of the Fibonacci sequence are 1, 1, 2, 3, and 5. Explain how these terms are related and generate the next five terms. Determine an expression for generating any term, F_n, in the sequence.

B. French mathematician Edouard Lucas (1842–91) named the sequence in the rabbit problem "the Fibonacci sequence." He studied the related sequence 1, 3, 4, ... , whose terms are generated in the same way as the Fibonacci sequence. Generate the next five terms of the Lucas sequence.

C. Starting with the Fibonacci sequence, create a new sequence by adding terms that are two apart. The first four terms are shown.

Fibonacci	1	1	2	3	5	8
New		1 + 2	1 + 3	2 + 5	3 + 8	

Repeat this process with the Lucas sequence. How are these new sequences related to the Fibonacci and Lucas sequences?

D. Determine the ratios of consecutive terms in the Fibonacci sequence. The first three ratios are shown.

$$\frac{F_2}{F_1} = \frac{1}{1} = 1, \qquad \frac{F_3}{F_2} = \frac{2}{1} = 2, \qquad \frac{F_4}{F_3} = \frac{3}{2} = 1.5$$

What happens to the ratios if you continue the process? What happens if you repeat this process with the Lucas sequence? Based on your answers, how are the Fibonacci and Lucas sequences related to a geometric sequence?

E. Starting with the Fibonacci sequence, create two new sequences as shown.

Fibonacci	1	1	2	3	
New 1	1 x 1	1 x 1	2 x 2	3 x 3	
New 2	1 x 2	1 x 3	2 x 5	3 x 8	

The first new sequence is the squares of the Fibonacci terms. The second is the products of Fibonacci terms that are two apart. How are these two sequences related? What happens if you repeat this process with the Lucas sequence?

F. Create a new sequence by multiplying a Fibonacci number by a Lucas number from the same position. How is this new sequence related to the Fibonacci sequence?

Reflecting

G. How are the Fibonacci and Lucas sequences similar? different?

H. Although the Fibonacci and Lucas sequences have different starting values, they share the same relationship between consecutive terms, and they have many similar properties. What properties do you think different sequences with the same relationship between consecutive terms have? How would you check your conjecture?

I. From part D, the Fibonacci and Lucas sequences are closely related to a geometric sequence. How are these sequences similar? different?

In Summary

Key Ideas

- The Fibonacci sequence is defined by the recursive formula $t_1 = 1$, $t_2 = 1$, $t_n = t_{n-1} + t_{n-2}$, where $n \in \mathbf{N}$ and $n > 2$. This sequence models the number of petals on many kinds of flowers, the number of spirals on a pineapple, and the number of spirals of seeds on a sunflower head, among other naturally occurring phenomena.
- The Lucas sequence is defined by the recursive formula $t_1 = 1$, $t_2 = 3$, $t_n = t_{n-1} + t_{n-2}$, where $n \in \mathbf{N}$ and $n > 2$, and has many of the properties of the Fibonacci sequence.

Need to Know

- In a recursive sequence, the terms depend on one or more of the previous terms.
- Two different sequences with the same relationship between consecutive terms have similar properties.

FURTHER Your Understanding

1. Pick any two numbers and use the same relationship between consecutive terms as the Fibonacci and Lucas sequences to generate a new sequence. What properties does this new sequence share with the Fibonacci and Lucas sequences?

2. Since the ratios of consecutive terms of the Fibonacci and Lucas sequences are *almost* constant, these sequences are similar to a geometric sequence. Substitute the general term for a geometric sequence, $t_n = ar^{n-1}$, into the recursive formulas for the Fibonacci and Lucas sequences, and solve for r. How does this value of r relate to what you found in part D?

3. A sequence is defined by the recursive formula $t_1 = 1$, $t_2 = 5$, $t_n = t_{n-1} + 2t_{n-2}$, where $n \in \mathbf{N}$ and $n > 2$.

a) Generate the first 10 terms.

b) Calculate the ratios of consecutive terms. What happens to the ratios?

c) Develop a formula for the general term.

Tech | *Support*

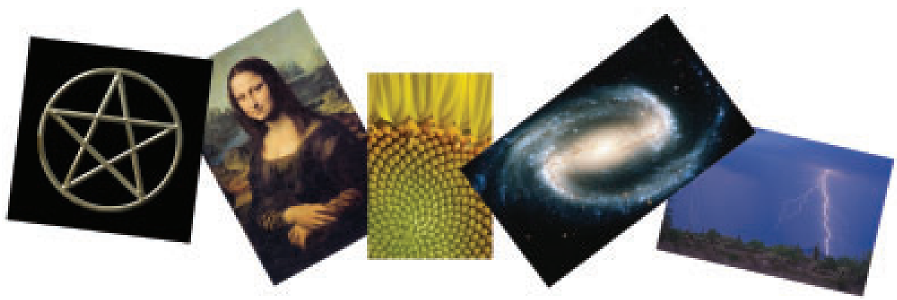
For help using a graphing calculator to generate sequences using recursive formulas, see Technical Appendix, B-16.

Curious Math

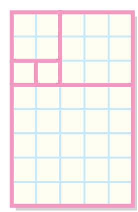
The Golden Ratio

The golden ratio (symbolized by ϕ, Greek letter phi) was known to the ancient Greeks. Euclid defined the golden ratio by a point C on a line segment AB such that $\phi = \frac{AC}{CB} = \frac{AB}{CB}$.

The golden ratio, like the Fibonacci sequence, seems to pop up in unexpected places. The ancient Greeks thought that it defined the most pleasing ratio to the eye, so they used it in their architecture. Artists have been known to incorporate the golden ratio into their works. It has even received some exposure in an episode of the crime series NUMB3RS, as well as in the movie and book *The Da Vinci Code*.



Human works aren't the only places where the golden ratio occurs. The ratio of certain proportions in the human body are close to the golden ratio, and spirals in seed heads of flowers can be expressed using the golden ratio.



- On a piece of graph paper, trace a 1×1 square.
- Draw another 1×1 square touching the left side of the first square.
- On top of these two squares, draw a 2×2 square.
- On the right side of your picture, draw a 3×3 square touching one of the 1×1 squares and the 2×2 square.
- Below your picture, draw a 5×5 square touching both 1×1 squares and the 3×3 square.
- Repeat this process of adding squares around the picture, alternating directions left, up, right, down, and so on. The start of the spiral is shown at the right.

1. How is this spiral related to the Fibonacci sequence and the golden ratio?