7.2 Geometric Sequences

YOU WILL NEED

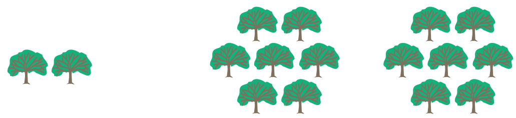
- graphing calculator
- graph paper

GOAL

Recognize the characteristics of geometric sequences and express the general terms in a variety of ways.

INVESTIGATE the Math

A local conservation group set up a challenge to get trees planted in a community. The challenge involves each person planting a tree and signing up seven other people to each do the same. Denise and Lise both initially accepted the challenge.



? If the pattern continues, how many trees will be planted at the 10th stage?

geometric sequence

a sequence that has the same ratio, the **common ratio**, between any pair of consecutive terms

A. Create the first five terms of the geometric sequence that represents the number of trees planted at each stage.

B. How is each term of this recursive sequence related to the previous term?

C. Use a graphing calculator to graph the term (number of trees planted) versus stage number. What type of relation is this?

D. Determine a formula for the general term of the sequence.

E. Use the general term to calculate the 10th term.

Reflecting

F. The tree-planting sequence is a geometric sequence. Another geometric sequence is 1 000 000, 500 000, 250 000, 125 000, ... How are the two sequences similar? Different?

G. How is the general term of a geometric sequence related to the equation of its graph?

H. A recursive formula for the tree-planting sequence is $t_1 = 2, t_n = 7t_{n-1}$, where $n \in \mathbf{N}$ and $n > 1$. How is this recursive formula related to the characteristics of this geometric sequence?

APPLY the Math

EXAMPLE 1 Connecting a specific term to the general term of a geometric sequence

a) Determine the 13th term of a geometric sequence if the first term is 9 and the common ratio is 2.
b) State a formula that defines each term of any geometric sequence.

Leo's Solution: Using a Recursive Formula

a)

n	1	2	3	4	5	6	7
t_n	9	18	36	72	144	288	576

n	8	9	10	11	12	13
t_n	1152	2304	4608	9216	18 432	36 864

I knew that the sequence is geometric so the terms increase by the same multiple each time. I made a table starting with the first term, and I multiplied each term by 2 to get the next term until I got the 13th term.

The 13th term is 36 864.

b) $a, ar, (ar)r, (ar^2)r, \ldots$

To get the terms of any geometric sequence, I would multiply the previous term by r each time, where a is the first term.

Recursive formula:

$t_1 = a, t_n = rt_{n-1}$, where $n \in \mathbf{N}$ and $n > 1$

Tamara's Solution: Using Powers of *r*

a) $a = 9$

$r = 2$

I knew that the sequence is geometric with first term 9 and common ratio 2.

$t_{13} = 9 \times 2^{12}$

$= 36\ 864$

To get the 13th term, I started with the first term. Then I multiplied the common ratio 12 times.

The 13th term is 36 864.

b) $a, ar, (ar)r, (ar^2)r, \ldots$

$= a, ar, ar^2, ar^3, \ldots$

I wrote a geometric sequence using a general first term, a, and a common ratio, r. I simplified the terms.

General term:

$t_n = ar^{n-1}$ or $f(n) = ar^{n-1}$

Each time I multiplied by r, the result was one less than the position number. I recognized this as an exponential function, so I knew that I had a formula for the general term.

Geometric sequences can be used to model problems that involve increases or decreases that change exponentially.

EXAMPLE 2 Solving a problem by using a geometric sequence

A company has 3 kg of radioactive material that must be stored until it becomes safe to the environment. After one year, 95% of the radioactive material remains. How much radioactive material will be left after 100 years?

Jacob's Solution

$$3, 3 \times 0.95, (3 \times 0.95) \times 0.95, (3 \times 0.95^2) \times 0.95, \ldots$$
$$= 3, 3 \times 0.95, 3 \times 0.95^2, 3 \times 0.95^3, \ldots$$

Every year, 95% of the radioactive material remains. I represented the amount of radioactive material as a sequence. The terms show the amounts in each year.

$$a = 3$$
$$r = 0.95$$

The sequence is geometric with first term 3 and common ratio 0.95.

$$f(n) = ar^{n-1}$$

I wrote the formula for the general term.

$$f(100) = 3 \times 0.95^{100-1}$$
$$= 3 \times 0.95^{99}$$
$$\doteq 0.019$$

I needed to determine the value of $f(n)$ when $n = 100$. So I substituted $a = 3$, $r = 0.95$, and $n = 100$ into the formula.

After the 100th year, there will be about 19 g of radioactive material left.

EXAMPLE 3 Selecting a strategy to determine the number of terms in a geometric sequence

How many terms are in the geometric sequence 52 612 659, 17 537 553, ... , 11?

Suzie's Solution

$$a = 52\,612\,659$$
$$r = \frac{17\,537\,553}{52\,612\,659} = \frac{1}{3}$$

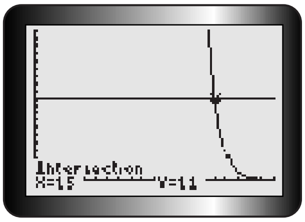
I knew that the sequence is geometric with first term 52 612 659. I calculated the common ratio by dividing t_2 by t_1.

$$f(n) = ar^{n-1}$$

I wrote the formula for the general term of a geometric sequence.

$$11 = 52\,612\,659 \times \left(\frac{1}{3}\right)^{n-1}$$

The last term of the sequence is 11, so its position number will be equal to the number of terms in the sequence. I determined the value of n when $f(n) = 11$ by substituting $a = 52\,612\,659$, $r = \frac{1}{3}$, and $f(n) = 11$ into the formula.



Instead of using guess and check to determine n, I graphed the functions Y1 = 52 612 659(1/3)^(X−1) and Y2 = 11 using my graphing calculator. Then I found the point of intersection. The x-coordinate represents the number of terms in the sequence.

There are 15 terms in the geometric sequence.

Tech *Support*

For help using a graphing calculator to determine the point of intersection of two functions, see Technical Appendix, B-12.

In Summary

Key Idea

- A geometric sequence is a recursive sequence in which new terms are created by multiplying the previous term by the same value (the common ratio) each time. For example, 2, 6, 18, 54, ... is increasing with a common ratio of 3,

  $\times 3 \times 3 \times 3$

$$\frac{t_2}{t_1} = \frac{6}{2} = 3$$
$$\frac{t_3}{t_2} = \frac{18}{6} = 3$$
$$\frac{t_4}{t_3} = \frac{54}{18} = 3$$
$$\vdots$$

and 144, 72, 36, 18, ... is decreasing with a common ratio of $\frac{1}{2}$.

$\times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2}$

$$\frac{t_2}{t_1} = \frac{72}{144} = \frac{1}{2}$$
$$\frac{t_3}{t_2} = \frac{36}{72} = \frac{1}{2}$$
$$\frac{t_4}{t_3} = \frac{18}{36} = \frac{1}{2}$$
$$\vdots$$

If the common ratio is negative, the sequence has terms that alternate from positive to negative. For example, 5, −20, 80, −320, ... has a common ratio of −4.

$\times (-4) \times (-4) \times (-4)$

Need to Know

- A geometric sequence can be defined
  - by the general term $t_n = ar^{n-1}$,
  - recursively by $t_1 = a$, $t_n = rt_{n-1}$, where $n > 1$, or
  - by a discrete exponential function $f(n) = ar^{n-1}$.

  In all cases, $n \in \mathbf{N}$, a is the first term, and r is the common ratio.

CHECK Your Understanding

1. Determine which sequences are geometric. For those that are, state the common ratio.

a) 15, 26, 37, 48, ...

b) 5, 15, 45, 135, ...

c) 3, 9, 81, 6561, ...

d) 6000, 3000, 1500, 750, 375, ...

2. State the general term and the recursive formula for each geometric sequence.

a) 9, 36, 144, ...

b) 625, 1250, 2500, ...

c) 10 125, 6750, 4500, ...

3. The 31st term of a geometric sequence is 123 and the 32nd term is 1107. What is the 33rd term?

4. What is the 10th term of the geometric sequence 1 813 985 280, 302 330 880, 50 388 480, ...?

PRACTISING

5. **i)** Determine whether each sequence is geometric.

ii) If a sequence is geometric, state the general term and the recursive formula.

a) 12, 24, 48, 96, ...

b) 1, 3, 7, 15, ...

c) 3, 6, 9, 12, ...

d) 5, −15, 45, −135, ...

e) 6, 7, 14, 15, ...

f) 125, 50, 20, 8, ...

6. For each geometric sequence, determine

K **i)** the general term **ii)** the recursive formula **iii)** t_6

a) 4, 20, 100, ...

b) −11, −22, −44, ...

c) 15, −60, 240, ...

d) 896, 448, 224, ...

e) $6, 2, \frac{2}{3}, \ldots$

f) 1, 0.2, 0.04, ...

7. **i)** Determine whether each sequence is arithmetic, geometric, or neither.

ii) If a sequence is arithmetic or geometric, state the general term.

a) 9, 13, 17, 21, ...

b) 7, −21, 63, −189, ...

c) 18, −18, 18, −18, ...

d) 31, 32, 34, 37, ...

e) 29, 19, 9, −1, ...

f) 128, 96, 72, 54, ...

8. Determine the recursive formula and the general term for the geometric sequence in which

a) the first term is 19 and the common ratio is 5

b) $t_1 = -9$ and $r = -4$

c) the first term is 144 and the second term is 36

d) $t_1 = 900$ and $r = \frac{1}{6}$

9. **i)** Determine whether each recursive formula defines a geometric sequence, where $n \in \mathbf{N}$.

ii) If the sequence is geometric, state the first five terms and the common ratio.

a) $t_1 = 18, t_n = \left(\frac{2}{3}\right)^{n-1} t_{n-1}$, where $n > 1$

b) $t_1 = -8, t_n = -3t_{n-1}$, where $n > 1$

c) $t_1 = 123, t_n = \frac{t_{n-1}}{3}$, where $n > 1$

d) $t_1 = 10, t_2 = 20, t_n = 4t_{n-2}$, where $n > 2$

10. **i)** Determine whether each general term defines a geometric sequence, where $n \in \mathbf{N}$.

ii) If the sequence is geometric, state the first five terms and the common ratio.

a) $t_n = 4^n$

b) $t_n = 3^n + 5$

c) $f(n) = n^2 - 13n + 8$

d) $t_n = 7 \times (-5)^{n-4}$

e) $f(n) = \frac{2}{3n + 1}$

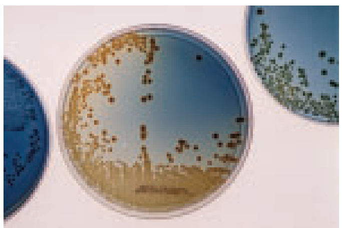
f) $f(n) = \frac{11}{13^n}$

11. The 5th term of a geometric sequence is 45 and the 8th term is 360. Determine the 20th term.

12. A doctor makes observations of a bacterial culture at fixed time intervals. The table below shows his first four observations. If the pattern continues, how many bacteria will be present at the 9th observation?

A

Observation	Number of Bacteria
1	5 120
2	7 680
3	11 520
4	17 280



13. Sam invested \$5000 in a GIC earning 8% compound interest per year. The interest gets added to the amount invested, so the next year Sam gets interest on the interest already earned, as well as on the original amount. How much will Sam's investment be worth at the end of 10 years?

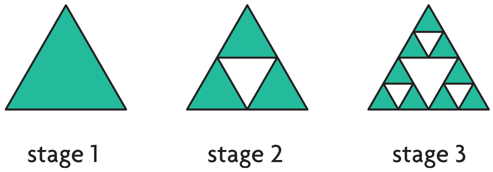
14. A certain antibiotic reduces the number of bacteria in your body by 10% each dose.

a) If four doses of the antibiotic are taken, what percent of the original bacterial population is left?

b) Biologists have determined that when a person has a bacterial infection, if the bacterial level can be reduced to 5% of its initial population, the person can fight off the infection. How many doses must be administered to reduce the bacterial population to the desired level?

15. You are given the 5th and 7th terms of a geometric sequence. Is it possible to determine the 29th term *without* finding the general term? If so, describe how you would do it.

16. The Sierpinski gasket is a fractal created from an equilateral triangle. At each stage, the "middle" is cut out of each remaining equilateral triangle. The first three stages are shown.
T



a) If the process continues indefinitely, the stages get closer to the Sierpinski gasket. How many shaded triangles would be present in the sixth stage?
b) If the triangle in the first stage has an area of 80 cm^2, what is the area of the shaded portion of the 20th stage?

17. In what ways are arithmetic and geometric sequences similar? Different?
C

Extending

18. Given the geometric sequence with $t_1 = 1$ and $r = \frac{1}{2}$, calculate the sum of the first 1, 2, 3, and 4 terms. What would happen to the sum if you added more and more terms?

19. Determine the 10th term of the sequence 3, 10, 28, 72, 176, ... State the general term.

20. Is it possible for the first three terms of an arithmetic sequence to be equal to the first three terms of a geometric sequence? If so, provide an example.

21. Create an arithmetic sequence such that some of its terms form a geometric sequence. How is the geometric sequence related to the arithmetic sequence?

22. A square has a side length of 12 cm. The midpoints of the square are joined creating a smaller square and four triangles. If you continue this process, what will be the total area of the shaded region in stage 6?