7.1 Arithmetic Sequences

YOU WILL NEED

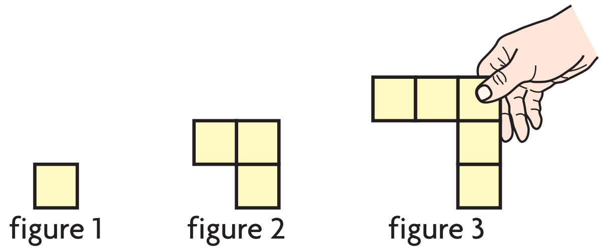
- linking cubes
- graphing calculator or graph paper
- spreadsheet software

GOAL

Recognize the characteristics of arithmetic sequences, and express the general terms in a variety of ways.

INVESTIGATE the Math

Chris used linking cubes to create different shapes. The first three shapes are shown. He wrote the **sequence** that represents the number of cubes in each shape.



? How many linking cubes are there in the 100th figure?

A. Create the next three **terms** of Chris's **arithmetic sequence.**

B. How is each term of this **recursive sequence** related to the previous term?

C. Construct a graph of term (number of cubes) versus figure number. What type of relation is this?

D. Determine a formula for the **general term** of the sequence.

E. Use the general term to calculate the 100th term.

Reflecting

F. Chris's sequence is an arithmetic sequence. Another arithmetic sequence is 635, 630, 625, 620, 615, ... How are the two sequences similar? Different?

G. How does the definition of an arithmetic sequence help you predict the shape of the graph of the sequence?

H. A **recursive formula** for Chris's sequence is $t_1 = 1$, $t_n = t_{n-1} + 2$, where $n \in \mathbf{N}$ and $n > 1$. How is this recursive formula related to the characteristics of Chris's arithmetic sequence?

sequence
an ordered list of numbers

term
a number in a sequence. Subscripts are usually used to identify the positions of the terms.

arithmetic sequence
a sequence that has the same difference, the **common difference**, between any pair of consecutive terms

recursive sequence
a sequence for which one term (or more) is given and each successive term is determined from the previous term(s)

general term
a formula, labelled t_n, that expresses each term of a sequence as a function of its position. For example, if the general term is $t_n = 2n$, then to calculate the 12th term (t_{12}), substitute $n = 12$.

$$\begin{aligned} t_{12} &= 2(12) \\ &= 24 \end{aligned}$$

recursive formula
a formula relating the general term of a sequence to the previous term(s)

APPLY the Math

EXAMPLE 1 Representing the general term of an arithmetic sequence

a) Determine a formula that defines the arithmetic sequence 3, 12, 21, 30, ...
b) State a formula that defines each term of any arithmetic sequence.

Wanda's Solution: Using Differences

a) $12 - 3 = 9$ ← I knew that the sequence is arithmetic, so the terms increase by the same amount. I subtracted t_1 from t_2 to determine the common difference.

$t_n = 3 + (n - 1)(9)$
$= 3 + 9n - 9$
$= 9n - 6$

← I wrote this sequence as 3, 3 + 9, 3 + 2(9), 3 + 3(9), ... Each multiple of 9 that I added was one less than the position number. So for the nth term, I needed to add $(n - 1)$ 9s.

The general term is $t_n = 9n - 6$.

b) $a, a + d, (a + d) + d, (a + 2d) + d, \ldots$
$= a, a + d, a + 2d, a + 3d, \ldots$

← I wrote an arithmetic sequence using a general first term, a, and a common difference, d. I simplified by collecting like terms.

General term:

$t_n = a + (n - 1)d$ ← Each multiple of d that I added was one less than the position number. So I knew that I had a formula for the general term.

Nathan's Solution: Using Multiples of 9

a) $12 = 3 + 9$ ← Since the sequence is arithmetic, to get each new term, I added 9 to the previous term.

$21 = 12 + 9$

$30 = 21 + 9$

$t_n = 9n$ ← Since I added 9 each time, I thought about the sequence of multiples of 9 because each term of that sequence goes up by 9s.

9, 18, 27, 36, ... ← Each term of my sequence is 6 less than the term in the same position in the sequence of multiples of 9.

3, 12, 21, 30, ...

The general term is $t_n = 9n - 6$. ← The general term of the sequence of multiples of 9 is $9n$, so I subtracted 6 to get the general term of my sequence.

b) $t_n = nd$

$d, 2d, 3d, 4d, \ldots$

Since I added the common difference *d* each time, I thought about the sequence of multiples of *d*.

$d + (a - d), 2d + (a - d), 3d + (a - d), 4d + (a - d), \ldots$

But the first term of my sequence was *a*, not *d*. So to get my sequence, I had to add *a* to, and subtract *d* from, each term of the sequence of multiples of *d*.

General term:

$$t_n = nd + (a - d)$$
$$= a + nd - d$$
$$= a + (n - 1)d$$

I simplified to get the general term.

Tina's Solution: Using a Recursive Formula

a) $12 = 3 + 9$

$21 = 12 + 9$

$30 = 21 + 9$

Since the sequence is arithmetic, to get each new term, I added 9 to the previous term.

The recursive formula is
$t_1 = 3, t_n = t_{n-1} + 9$, where $n \in \mathbf{N}$ and $n > 1$.

Since I added 9 each time, I expressed the general term of the sequence using a recursive formula.

b) $a, a + d, a + 2d, a + 3d, \ldots$

To get the terms of any arithmetic sequence, I would add *d* to the previous term each time, where *a* is the first term.

Recursive formula:

$t_1 = a, t_n = t_{n-1} + d$, where $n \in \mathbf{N}$ and $n > 1$

Once you know the general term of an arithmetic sequence, you can use it to determine *any* term in the sequence.

EXAMPLE 2 Connecting a specific term to the general term of an arithmetic sequence

What is the 33rd term of the sequence 18, 11, 4, −3, ...?

David's Solution: Using Differences and the General Term

$11 - 18 = -7$

$4 - 11 = -7$

$-3 - 4 = -7$

I subtracted consecutive terms and found that each term is 7 less than the previous term. So the sequence is arithmetic.

$a = 18, d = -7$

$t_n = a + (n - 1)d$

The first term of the sequence is 18. Since the terms are decreasing, the common difference is −7.

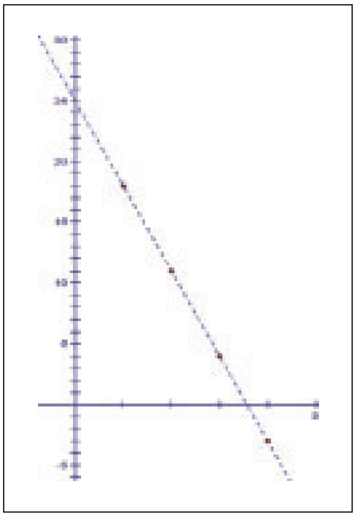
$t_{33} = 18 + (33 - 1)(-7)$

$= -206$

The 33rd term is −206.

I substituted these numbers into the formula for the general term of an arithmetic sequence. To get the 33rd term, I let $n = 33$.

Leila's Solution: Using a Graph and Function Notation



I represented the sequence as a function using ordered pairs with the term number (n) as the x-coordinate and the term (t_n) as the y-coordinate.

n	t_n	1st Differences
1	18	
		−7
2	11	
		−7
3	4	
		−7
4	−3	

The 1st differences are constant so these points lie on a line.

Since $n \in \mathbf{N}$ and the terms lie on this line, I used a dashed line to connect the points.

$f(x) = -7x + 25$

The slope of the line is $m = -7$ and the y-intercept is $b = 25$. I used this information to write the function that describes the line.

The y-intercept corresponds to the term t_0 but it is *not* a term of the sequence since $x \in \mathbf{N}$.

$f(33) = -7(33) + 25$

$= -206$

The 33rd term is −206.

To get the 33rd term of the sequence, I substituted $x = 33$ into the equation $f(x) = -7x + 25$.

Communication *Tip*

A dashed line on a graph indicates that the x-coordinates of the points on the line are natural numbers.

Arithmetic sequences can be used to model problems that involve increases or decreases that occur at a constant rate.

EXAMPLE 3 Representing an arithmetic sequence

Terry invests $300 in a GIC (guaranteed investment certificate) that pays 6% simple interest per year. When will his investment be worth $732?

Philip's Solution: Using a Spreadsheet

6% of $300 = $18

Terry earns 6% of $300, which is $18 interest per year.

	A	B
1	Year	Terry's $
2	1	$300.00
3	2	= B2+18
4		

I set up a spreadsheet. In one column I entered the year number, and in the other column, I entered the amount. I set up a formula to increase the amount by $18 per year.

	A	B
1	Year	Terry's $
2	1	$300.00
3	2	$318.00
4	3	$336.00
5	4	$354.00
6	5	$372.00
7	6	$390.00
8	7	$408.00
9	8	$426.00
10	9	$444.00
11	10	$462.00
12	11	$480.00
13	12	$498.00
14	13	$516.00
15	14	$534.00
16	15	$552.00
17	16	$570.00
18	17	$588.00
19	18	$606.00
20	19	$624.00
21	20	$642.00
22	21	$660.00
23	22	$678.00
24	23	$696.00
25	24	$714.00
26	25	$732.00

I used the spreadsheet to continue the pattern until the amount reached $732.

From the spreadsheet, Terry's investment will be worth $732 at the beginning of the 25th year.

Tech | *Support*

For help using a spreadsheet to enter values and formulas, fill down, and fill right, see Technical Appendix, B-21.

Jamie's Solution: Using the General Term

$t_n = a + (n - 1)d$

$t_n = 300 + (n - 1)(18)$ ← Terry earns 6% of \$300, or \$18 interest, per year. So his investment increases by \$18/year. This is an arithmetic sequence, where $a = 300$ and $d = 18$.

$732 = 300 + (n - 1)(18)$ ← I needed to determine when $t_n = 732$.

$732 = 300 + 18n - 18$ ← I solved for n.

$732 - 300 + 18 = 18n$

$450 = 18n$

$25 = n$

Terry's investment will be worth \$732 in the 25th year.

Suzie's Solution: Using Reasoning

$a = 300, d = 18$ ← Terry earns 6% of \$300 = \$18 interest per year. So the amount at the start of each year will form an arithmetic sequence.

$732 - 300 = 432$ ← I calculated the difference between the starting and ending values to know how much interest was earned.

$432 \div 18 = 24$ ← I divided by the amount of interest paid per year to determine how many interest payments were made.

The investment will be worth \$732 at the beginning of the 25th year. ← Since interest was paid every year except the first year, \$732 must occur in the 25th year.

If you know two terms of an arithmetic sequence, you can determine *any* term in the sequence.

EXAMPLE 4 Solving an arithmetic sequence problem

The 7th term of an arithmetic sequence is 53 and the 11th term is 97. Determine the 100th term.

Tanya's Solution: Using Reasoning

$$t_{11} - t_7 = 97 - 53$$
$$= 44$$

I knew that the sequence is arithmetic, so the terms increase by the same amount each time.

$$4d = 44$$
$$d = 11$$

There are four differences to go from t_7 to t_{11}. So I divided 44 by 4 to get the common difference.

$$t_{100} = 97 + 89 \times 11$$
$$= 97 + 979$$
$$= 1076$$

Since the common difference is 11, I knew that to get the 100th term, I would have to add it to t_{11} 89 times.

The 100th term is 1076.

Deepak's Solution: Using Algebra

$$t_n = a + (n - 1)d$$

I knew that the sequence is arithmetic, so I wrote the formula for the general term.

t_7

$$53 = a + (7 - 1)d$$
$$53 = a + 6d$$

t_{11}

$$97 = a + (11 - 1)d$$
$$97 = a + 10d$$

For the 7th term, I substituted $t_7 = 53$ and $n = 7$ into the general term. For the 11th term, I substituted $t_{11} = 97$ and $n = 11$. Since both equations describe terms of the same arithmetic sequence, *a* and *d* are the same in both equations.

$$97 = a + 10d$$
$$-53 = -(a + 6d)$$
$$44 = 4d$$
$$11 = d$$

The equations for t_7 and t_{11} represent a linear system. To solve for *d*, I subtracted the equation for t_7 from the equation for t_{11}.

$$53 = a + 6(11)$$
$$53 = a + 66$$
$$-13 = a$$

To solve for a, I substituted $d = 11$ into the equation for t_7.

$$t_n = a + (n - 1)d$$
$$t_{100} = -13 + (100 - 1)(11)$$
$$= 1076$$

To get the 100th term, I substituted $a = -13$, $d = 11$, and $n = 100$ into the formula for the general term.

The 100th term is 1076.

In Summary

Key Ideas

- Every sequence is a discrete function. Since each term is identified by its position in the list (1st, 2nd, and so on), the domain is the set of natural numbers, **N** = {1, 2, 3, ...}. The range is the set of all the terms of the sequence.
  For example, 4, 12, 20, 28, ... (4 is the 1st term, 12 is the 2nd term)

  Domain: {1, 2, 3, 4, ...}
  Range: {4, 12, 20, 28, ...}

- An arithmetic sequence is a recursive sequence in which new terms are created by adding the same value (the common difference) each time.
  For example, 2, 6, 10, 14, ... is increasing with a common difference of 4, (+4 +4 +4)

  $t_2 - t_1 = 6 - 2 = 4$
  $t_3 - t_2 = 10 - 6 = 4$
  $t_4 - t_3 = 14 - 10 = 4$
  ⋮

  and 9, 6, 3, 0, ... is decreasing with a common difference of -3. (−3 −3 −3)

  $t_2 - t_1 = 6 - 9 = -3$
  $t_3 - t_2 = 3 - 6 = -3$
  $t_4 - t_3 = 0 - 3 = -3$
  ⋮

Need to Know

- An arithmetic sequence can be defined
  - by the general term $t_n = a + (n - 1)d$,
  - recursively by $t_1 = a$, $t_n = t_{n-1} + d$, where $n > 1$, or
  - by a discrete linear function $f(n) = dn + b$, where $b = t_0 = a - d$.

  In all cases, $n \in$ **N**, a is the first term, and d is the common difference.

CHECK Your Understanding

1. Determine which sequences are arithmetic. For those that are, state the common difference.

a) 1, 5, 9, 13, 17, ...

b) 3, 7, 13, 17, 23, 27, ...

c) 3, 6, 12, 24, ...

d) 59, 48, 37, 26, 15, ...

2. State the general term and the recursive formula for each arithmetic sequence.

a) 28, 42, 56, ... **b)** 53, 49, 45, ... **c)** $-1, -111, -221, \ldots$

3. The 10th term of an arithmetic sequence is 29 and the 11th term is 41. What is the 12th term?

4. What is the 15th term of the arithmetic sequence 85, 102, 119, ...?

PRACTISING

5. **i)** Determine whether each sequence is arithmetic.

ii) If a sequence is arithmetic, state the general term and the recursive formula.

a) 8, 11, 14, 17, ...

b) 15, 16, 18, 19, ...

c) 13, 31, 13, 31, ...

d) 3, 6, 12, 24, ...

e) 23, 34, 45, 56, ...

f) $\frac{1}{6}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \ldots$

6. Determine the recursive formula and the general term for the arithmetic sequence in which

a) the first term is 19 and consecutive terms increase by 8

b) $t_1 = 4$ and consecutive terms decrease by 5

c) the first term is 21 and the second term is 26

d) $t_4 = 35$ and consecutive terms decrease by 12

7. **i)** Determine whether each recursive formula defines an arithmetic sequence, where $n \in \mathbf{N}$ and $n > 1$.

ii) If the sequence is arithmetic, state the first five terms and the common difference.

a) $t_1 = 13, t_n = 14 + t_{n-1}$

b) $t_1 = 5, t_n = 3t_{n-1}$

c) $t_1 = 4, t_n = t_{n-1} + n - 1$

d) $t_1 = 1, t_n = 2t_{n-1} - n + 2$

8. For each arithmetic sequence, determine

K **i)** the general term **ii)** the recursive formula **iii)** t_{11}

a) 35, 40, 45, ...

b) 31, 20, 9, ...

c) $-29, -41, -53, \ldots$

d) 11, 11, 11, ...

e) $1, \frac{6}{5}, \frac{7}{5}, \ldots$

f) 0.4, 0.57, 0.74, ...

9. **i)** Determine whether each general term defines an arithmetic sequence.
ii) If the sequence is arithmetic, state the first five terms and the common difference.

a) $t_n = 8 - 2n$

b) $t_n = n^2 - 3n + 7$

c) $f(n) = \frac{1}{4}n + \frac{1}{2}$

d) $f(n) = \frac{2n + 5}{7 - 3n}$

10. **A** An opera house has 27 seats in the first row, 34 seats in the second row, 41 seats in the third row, and so on. The last row has 181 seats.
a) How many seats are in the 10th row?
b) How many rows of seats are in the opera house?

11. Janice gets a job and starts out earning \$9.25/h. Her boss promises her a raise of \$0.15/h after each month of work. When will Janice start earning at least twice her starting wage?

12. Phil invests \$5000 in a high-interest savings account and earns 3.5% simple interest per year. How long will he have to leave his money in the account if he wants to have \$7800?

13. Determine the number of terms in each arithmetic sequence.
a) 7, 9, 11, 13, ... , 63
b) −20, −25, −30, −35, ... , −205
c) 31, 27, 23, 19, ... , −25
d) 9, 16, 23, 30, ... , 100
e) −33, −26, −19, −12, ... , 86
f) 28, 19, 10, 1, ... , −44

14. **T** You are given the 4th and 8th terms of a sequence. Describe how to determine the 100th term *without* finding the general term.

15. The 50th term of an arithmetic sequence is 238 and the 93rd term is 539. State the general term.

16. **C** Two terms of an arithmetic sequence are 20 and 50.
a) Create three different arithmetic sequences given these terms. Each of the three sequences should have a different first term and a different common difference.
b) How are the common differences related to the terms 20 and 50?

Extending

17. The first term of an arithmetic sequence is 13. Two other terms of the sequence are 37 and 73. The common difference between consecutive terms is an integer. Determine all possible values for the 100th term.

18. Create an arithmetic sequence that has $t_1 > 0$ and in which each term is greater than the previous term. Create a new sequence by picking, from the original sequence, the terms described by the sequence. (For example, for the sequence 3, 7, 11, 15, ... , you would choose the 3rd, 7th, 11th, 15th, ... terms of the original sequence as t_1, t_2, t_3, t_4, ... of your new sequence.) Is this new sequence always arithmetic?