6.7 Solving Problems Using Sinusoidal Models

GOAL

Solve problems related to real-world applications of sinusoidal functions.

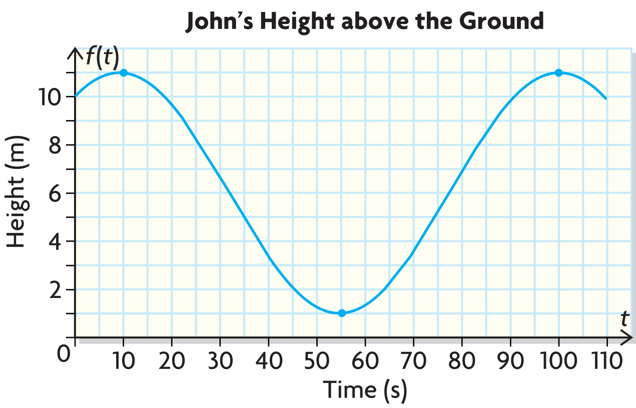
LEARN ABOUT the Math

A group of students is tracking a friend, John, who is riding a Ferris wheel. They know that John reaches the maximum height of 11 m at 10 s and then reaches the minimum height of 1 m at 55 s.

? How can you develop the equation of a sinusoidal function that models John's height above the ground to determine his height at 78 s?

EXAMPLE 1 Connecting the equation of a sinusoidal function to the situation

Justine's Solution



I plotted the two points I knew: (10, 11) and (55, 1).

Since it takes John 45 s to go from the highest point to the lowest, then it would take him 90 s to do one complete revolution and be back to a height of 11 m at 100 s.

I drew a smooth curve to connect the points to look like a wave.

Vertical translation: c

equation of the axis:

$$y = \frac{11 + 1}{2} = 6$$

$$c = 6$$

I found the equation of the axis by adding the maximum and minimum and dividing by 2. That gave me the vertical translation and the value of c.

Vertical stretch: a

amplitude $= 11 - 6 = 5$

$$a = 5$$

I found the amplitude by taking the maximum and subtracting the y-value for the equation of the axis. That gave me the vertical stretch and the value of a.

Horizontal compression: k

For the horizontal compression, I used the formula relating the period to k. The curve wasn't reflected, so k is positive.

$$\text{period} = \frac{360}{|k|}$$

$$k > 0, \text{ so the period} = \frac{360}{k}$$

$$90 = \frac{360}{k}$$

$$k = \frac{360}{90}$$

$$k = 4$$

Horizontal translation: d

If I use the cosine function, the first maximum is at $x = 0$. The first maximum of the new function is at $x = 10$. So there was a horizontal translation of 10. That gave me the value of d.

$d = 10$

$y = 5\cos(4(x - 10)^\circ) + 6$

I got the equation of the sinusoidal function by substituting the values I found into $y = a\cos(k(x - d)) + c$.

$$y = 5\cos(4(78 - 10)^\circ) + 6$$
$$= 5\cos 272^\circ + 6$$
$$y \doteq 5(0.035) + 6$$
$$\doteq 6.17 \text{ m}$$

Once I had the equation, I substituted $x = 78$, and solved for the height.

At 78 s, his height will be about 6.17 m.

The answer 6.17 m looks reasonable based on the graph.

Reflecting

A. If it took John 60 s instead of 90 s to complete one revolution, how would the sinusoidal function change? State the value and type of transformation associated with this change.

B. If the radius of the Ferris wheel remained the same but the axle of the wheel was 1 m higher, how would the sinusoidal function change? State the value and type of transformation associated with this change.

C. If both characteristics from parts A and B were changed, what would be the equation of the sinusoidal function describing John's height above the ground in terms of time?

APPLY the Math

EXAMPLE 2 Solving a problem involving a sinusoidal function

The top of a flagpole sways back and forth in high winds. The top sways 10 cm to the right $(+10 \text{ cm})$ and 10 cm to the left (-10 cm) of its resting position and moves back and forth 240 times every minute. At $t = 0$, the pole was momentarily at its resting position. Then it started moving to the right.

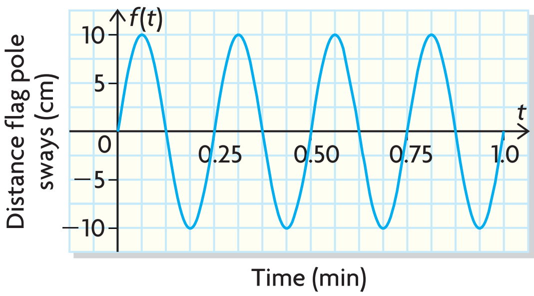
a) Determine the equation of a sinusoidal function that describes the distance the top of the pole is from its resting position in terms of time.

b) How does the situation affect the domain and range?

c) If the wind speed decreases slightly such that the sway of the top of the pole is reduced by 20%, what is the new equation of the sinusoidal function? Assume that the period remains the same.

Ryan's Solution

a)



I drew a graph where time is the independent variable, and the distance the top of the pole moves is the dependent variable.

The highest point on my graph will be 10, and the lowest will be −10.

I started at (0, 0) because the pole was at its resting position at $t = 0$.

$$\text{Number of sways each second} = \frac{240}{60} = 4$$

Since the pole sways back and forth 240 times in 60 s, the time to complete one sway must be 0.25 s. This is the period.

$$\text{period} = \frac{1}{4} = 0.25$$

Vertical translation: c

The axis is at $y = 0$. This gives the vertical translation.

equation of the axis: $y = 0$

so $c = 0$

Vertical stretch: a

I took the distance between a peak and the equation of the axis to get the amplitude.

amplitude $= 10$

so $a = 10$

Horizontal compression: k

I found the horizontal compression from the formula relating the period to the value of k.

$$\text{period} = \frac{360}{|k|}$$

$$k > 0$$

$$\text{period} = \frac{360}{|k|}$$

$$0.25 = \frac{360}{k}$$

$$k = \frac{360}{0.25}$$

$$k = 1440$$

The sine function:

I decided to use the sine function since this graph starts at $(0°, 0)$. Using the values of a and k, i determined the equation

$$y = 10\sin(1440x)°$$

Horizontal translation: d

For the cosine function, the horizontal translation is equal to the x-coordinate of any maximum, since the maximum of a cosine function is at 0. I used the x-coordinate of the first maximum of the new function. That maximum is at $t = \frac{1}{16}$.

$$d = \frac{1}{16}$$

$$y = 10\cos\left(1440\left(x - \frac{1}{16}\right)°\right)$$

I put all these transformations together to get the equation of the function.

b) For either function, the domain is restricted to positive values because the values represent the time elapsed. The range of each function depends on its amplitudes.

c) 80% of 10

If the sway is the only thing that's changing, then the amplitude is going to change on the graph.
If the sway is reduced by 20%, it's 80% of what it used to be. The amplitude will then change from 10 to 8.
The vertical stretch is 8.

$$= 0.80 \times 10$$

$$= 8$$

$$y = 8\cos\left(1440\left(x - \frac{1}{16}\right)°\right)$$

or $y = 8\sin(1440x)°$

In Summary

Key Idea

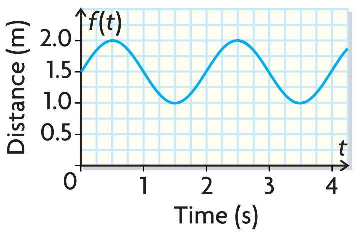
- Algebraic and graphical models of the sine and cosine functions can be used to solve a variety of real-world problems involving periodic behaviour.

Need to Know

- When you have a description of an event that can be modelled by a sinusoidal graph rather than data, it is useful to organize the information presented by drawing a rough sketch of the graph.
- You will have to determine the equation of the sinusoidal function by first calculating the period, amplitude, and equation of the axis. This information will help you determine the values of k, a, and c, respectively, in the equations $g(x) = a\sin(k(x-d)) + c$ and $h(x) = a\cos(k(x-d)) + c$.

CHECK Your Understanding

Distance Between the Tail Light and the Curb



1. The load on a trailer has shifted, causing the rear end of the trailer to swing left and right. The distance from one of the tail lights on the trailer to the curb varies sinusoidally with time. The graph models this behaviour.

a) What is the equation of the axis of the function, and what does it represent in this situation?

b) What is the amplitude of the function, and what does it represent in this situation?

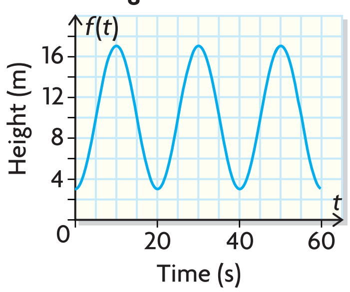
c) What is the period of the function, and what does it represent in this situation?

d) Determine the equation and the range of the sinusoidal function.

e) What are the domain and range of the function in terms of the situation?

f) How far is the tail light from the curve at $t = 3.2$ s?

Height above Ground



2. Don Quixote, a fictional character in a Spanish novel, attacked windmills because he thought they were giants. At one point, he got snagged by one of the blades and was hoisted into the air. The graph shows his height above ground in terms of time.

a) What is the equation of the axis of the function, and what does it represent in this situation?

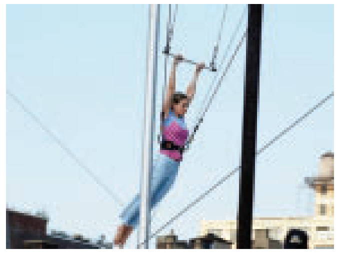
b) What is the amplitude of the function, and what does it represent in this situation?

c) What is the period of the function, and what does it represent in this situation?

d) If Don Quixote remains snagged for seven complete cycles, determine the domain and range of the function.

e) Determine the equation of the sinusoidal function.

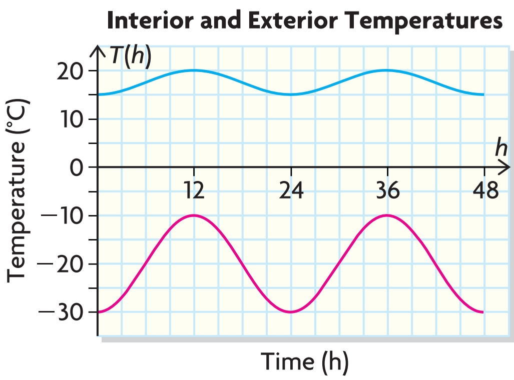
f) If the wind speed decreased, how would that affect the graph of the sinusoidal function?



3. Chantelle is swinging back and forth on a trapeze. Her distance from a vertical support beam in terms of time can be modelled by a sinusoidal function. At 1 s, she is the maximum distance from the beam, 12 m. At 3 s, she is the minimum distance from the beam, 4 m. Determine an equation of a sinusoidal function that describes Chantelle's distance from the vertical beam in relation to time.

PRACTISING

4. **K** The interior and exterior temperatures of an igloo were recorded over a 48 h period. The data were collected and plotted, and two curves were drawn through the appropriate points.



a) How are these curves similar? Explain how each of them might be related to this situation.
b) Describe the domain and range of each curve.
c) Assuming that the curves can be represented by sinusoidal functions, determine the equation of each function.

5. **A** Skyscrapers sway in high-wind conditions. In one case, at $t = 2$ s, the top floor of a building swayed 30 cm to the left (-30 cm), and at $t = 12$, the top floor swayed 30 cm to the right $(+30 \text{ cm})$ of its starting position.
a) What is the equation of a sinusoidal function that describes the motion of the building in terms of time?
b) Dampers, in the forms of large tanks of water, are often added to the top floors of skyscrapers to reduce the severity of the sways. If a damper is added to this building, it will reduce the sway (not the period) by 70%. What is the equation of the new function that describes the motion of the building in terms of time?

6. Milton is floating in an inner tube in a wave pool. He is 1.5 m from the bottom of the pool when he is at the trough of a wave. A stopwatch starts timing at this point. In 1.25 s, he is on the crest of the wave, 2.1 m from the bottom of the pool.
a) Determine the equation of the function that expresses Milton's distance from the bottom of the pool in terms of time.

b) What is the amplitude of the function, and what does it represent in this situation?

c) How far above the bottom of the pool is Milton at $t = 4$ s?

d) If data are collected for only 40 s, how many complete cycles of the sinusoidal function will there be?

e) If the period of the function changes to 3 s, what is the equation of this new function?

7. An oscilloscope hooked up to an alternating current (AC) circuit shows a sine curve. The device records the current in amperes (A) on the vertical axis and the time in seconds on the horizontal axis. At $t = 0$ s, the current reads its first maximum value of 4.5 A. At $t = \frac{1}{120}$ s, the current reads its first minimum value of -4.5 A. Determine the equation of the function that expresses the current in terms of time.

8. Candice is holding onto the end of a spring that is attached to a lead ball. As she moves her hand slightly up and down, the ball moves up and down. With a little concentration, she can repeatedly get the ball to reach a maximum height of 20 cm and a minimum height of 4 cm from the top of a surface. The first maximum height occurs at 0.2 s, and the first minimum height occurs at 0.6 s.

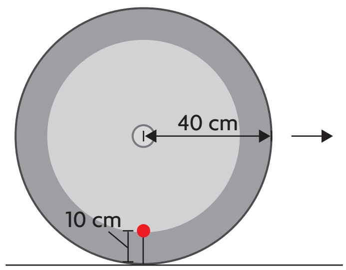
a) Determine the equation of the sinusoidal function that represents the height of the lead ball in terms of time.

b) Determine the domain and range of the function.

c) What is the equation of the axis, and what does it represent in this situation?

d) What is the height of the lead ball at 1.3 s?

9. A paintball is shot at a wheel of radius 40 cm. The paintball leaves a circular mark 10 cm from the outer edge of the wheel. As the wheel rolls, the mark moves in a circular motion.



a) Assuming that the paintball mark starts at its lowest point, determine the equation of the sinusoidal function that describes the height of the mark in terms of the distance the wheel travels.

b) If the wheel completes five revolutions before it stops, determine the domain and range of the sinusoidal function.

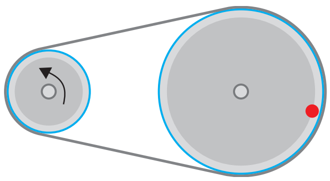
c) What is the height of the mark when the wheel has travelled 120 cm from its initial position?

10. The population of rabbits, $R(t)$, and the population of foxes, $F(t)$, in a given region are modelled by the functions $R(t) = 10\ 000 + 5000 \cos(15t)°$ and $F(t) = 1000 + 500 \sin(15t)°$, where t is the time in months. Referring to each graph, explain how the number of rabbits and the number of foxes are related.
T

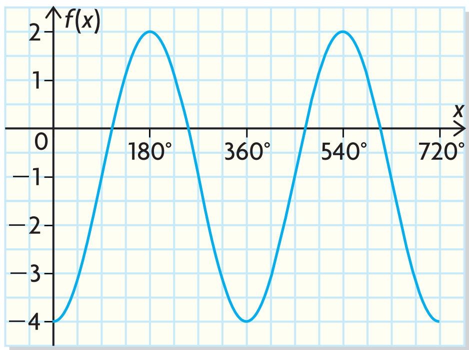
11. What information would you need to determine an algebraic or graphical model of a situation that could be modelled with a sinusoidal function?
C

Extending

12. Two pulleys are connected by a belt. Pulley A has a radius of 3 cm, and Pulley B has a radius of 6 cm. As Pulley A rotates, a drop of paint on the circumference of Pulley B rotates around the axle of Pulley B. Initially, the paint drop is 7 cm above the ground. Determine the equation of a sinusoidal function that describes the height of the drop of paint above the ground in terms of the rotation of Pulley A.



13. Examine the graph of the function $f(x)$.



a) Determine the equation of the function.
b) Evaluate $f(20)$.
c) If $f(x) = 2$, then which of the following is true for x?
- **i)** $180° + 360°k,\ k \in \mathbf{I}$
- **ii)** $360° + 180°k,\ k \in \mathbf{I}$
- **iii)** $90° + 180°k,\ k \in \mathbf{I}$
- **iv)** $270° + 360°k,\ k \in \mathbf{I}$

d) If $f(x) = -1$, then which of the following is true for x?
- **i)** $180° + 360°k,\ k \in \mathbf{I}$
- **ii)** $360° + 90°k,\ k \in \mathbf{I}$
- **iii)** $90° + 360°k,\ k \in \mathbf{I}$
- **iv)** $90° + 180°k,\ k \in \mathbf{I}$

14. Using graphing technology, determine x when $f(x) = 7$ for the function $f(x) = 4\cos(2x) + 3$ in the domain $\{x \in \mathbf{R} \mid 0° \leq x \leq 360°\}$.

Music

Pressing certain piano keys at the same time produces consonance, or pleasant sounds. Some combinations of keys produce dissonance, or unpleasant sounds.

When you strike a key, a string vibrates, causing the air to vibrate. This vibration of air produces a sound wave that your ear detects. The sound waves caused by striking various notes can be described by the functions in the table, where x is time in seconds and $f(x)$ is the displacement (or movement) of air molecules in micrometres $(1 \times 10^{-6}\ \text{m})$.

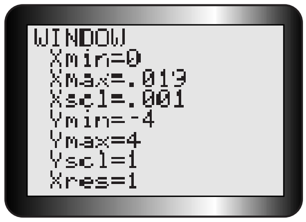
YOU WILL NEED

- graphing calculator

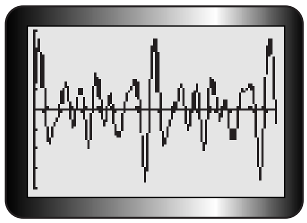
Equations for Notes (n.o. means next octave)

Note	Equation	Note	Equation	Note	Equation
A	$f(x) = \sin(158\,400x)°$	D	$f(x) = \sin(211\,427x)°$	G	$f(x) = \sin(282\,239x)°$
A#	$f(x) = \sin(167\,831x)°$	D#	$f(x) = \sin(224\,026x)°$	G#	$f(x) = \sin(299\,015x)°$
B	$f(x) = \sin(177\,806x)°$	E	$f(x) = \sin(237\,348x)°$	A n.o.	$f(x) = \sin(316\,800x)°$
C	$f(x) = \sin(188\,389x)°$	F	$f(x) = \sin(251\,465x)°$	B n.o.	$f(x) = \sin(355\,612x)°$
C#	$f(x) = \sin(199\,584x)°$	F#	$f(x) = \sin(266\,402x)°$	C n.o.	$f(x) = \sin(376\,777x)°$

One combination of notes is the A major chord, which is made up of A, C#, E, and A in the next octave. The sound can be modelled by graphing the sum of the equations for each note in Y1 using the WINDOW settings shown.



A major



1. Is the function for the A major chord periodic, sinusoidal, or both?
2. The C major chord is made up of C, E, G, and C in the next octave. Graph this function using your graphing calculator. Sketch the graph in your notebook. Compare the C major graph with the A major graph.
3. If you strike the keys A, B, C#, and F, the sound will be dissonance rather than consonance. Graph the function for this series of notes using your graphing calculator. Sketch the resulting curve. Compare with the C major and the A major graphs.
4. Graph and sketch each combination of notes below using your graphing calculator and the WINDOW settings shown above. Which combinations display consonance and which display dissonance?
   a) CC (C in first octave, C in next octave)
   b) CF
   c) CD
   d) CB (B in next octave)