5.7 The Cosine Law

GOAL

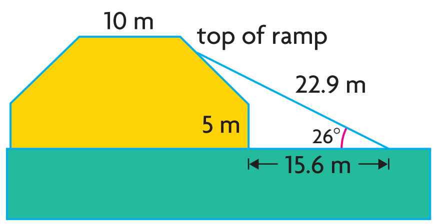
Solve two-dimensional problems by using the cosine law.

YOU WILL NEED

- dynamic geometry software (optional)

LEARN ABOUT the Math

A barn whose cross-section resembles half a regular octagon with a side length of 10 m needs some repairs to its roof. The roofers place a 22.9 m ramp against the side of the building, forming an angle of 26° with the ground. The ramp will be used to transport the materials needed for the repair. The base of the ramp is 15.6 m from the side of the building.

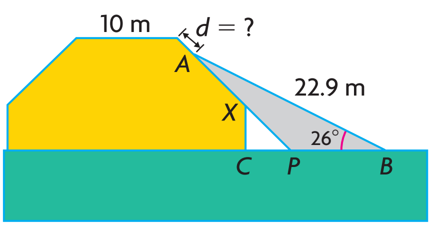


? How far, to the nearest tenth of a metre, is the top of the ramp from the flat roof of the building?

EXAMPLE 1 Using the cosine law to calculate an unknown length

Determine the distance from the top of the ramp to the roof by using the **cosine law**.

Tina's Solution

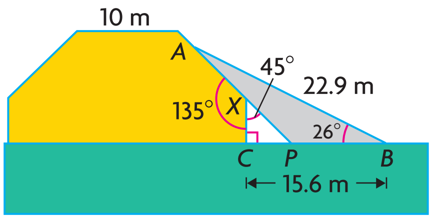


I labelled the top of the ramp *A* and the bottom of the ramp *B*. Then I drew a line from *A* along the sloped part of the building to *X* and extended the line to the ground at *P*. I labelled the point where the side of the building touches the ground *C*.

$$\angle AXC + \angle CXP = 180°$$
$$135° + \angle CXP = 180°$$
$$\angle CXP = 180° - 135°$$
$$= 45°$$

In $\triangle XCP$, $\angle C$ is 90°. Since the octagon is regular, each interior angle is 135°. So $\angle AXC$ is 135°. To determine $\angle CXP$, I subtracted 135° from 180°.

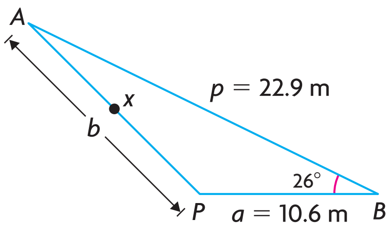
$\therefore$ $\triangle XCP$ is a 45° − 45° − 90° special triangle.



$$CP + PB = 15.6$$
$$5 + PB = 15.6$$
$$PB = 15.6 - 5$$
$$= 10.6 \text{ m}$$

From the given information, I knew that *XC* = 5 m, so *CP* = 5 m, since the triangle is isosceles.

I then subtracted *CP* from *CB* to determine the length of *PB*.



In △*APB*, I knew two side lengths and the contained angle formed by those sides. So I couldn't use the sine law to determine *AP*. I used the cosine law instead.

$$b^2 = a^2 + p^2 - 2ap\cos B$$
$$b^2 = (10.6)^2 + (22.9)^2 - 2(10.6)(22.9)\cos 26°$$
$$b^2 = 200.42 \text{ m}^2$$
$$b = \sqrt{200.42}$$
$$b \doteq 14.16 \text{ m}$$

I substituted the values of *a*, *p*, and ∠*B* into the formula. I calculated *b* by evaluating the right side of the equation and determining its square root.

$$XP = 5\sqrt{2}$$
$$AX + XP = b$$
$$AX + 5\sqrt{2} = 14.16$$
$$AX = 14.16 - 5\sqrt{2}$$
$$\doteq 7.09 \text{ m}$$

To determine the distance from the top of the ramp to the roof, I needed to calculate *AX* first. I knew that *XP* is a multiple of $\sqrt{2}$ because △*XCP* is a $45° - 45° - 90°$ special triangle. So I subtracted *XP* from *b* to determine *AX*.

$$\text{required distance} = 10 - AX$$
$$= 10 - 7.09$$
$$\doteq 2.9 \text{ m}$$

Then I subtracted *AX* from 10 m to get the distance from the top of the ramp to the roof.

The top of the ramp is about 2.9 m from the flat roof of the building.

Reflecting

A. Why did Tina draw line AP on her sketch as part of her solution?

B. Could Tina have used the sine law, instead of the cosine law, to solve the problem? Explain your reasoning.

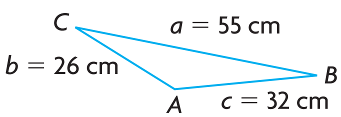
C. The Pythagorean theorem is a special case of the cosine law. What conditions would have to exist in a triangle in order for the cosine law to simplify to the Pythagorean theorem?

APPLY the Math

EXAMPLE 2 | Using the cosine law to determine an angle

In $\triangle ABC$, determine $\angle A$ to the nearest degree if $a = 55$ cm, $b = 26$ cm, and $c = 32$ cm.

Claudio's Solution



I drew a diagram of the triangle and labelled all sides.

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Since I knew all three sides (SSS) but no angles, I couldn't use the sine law. So I used the cosine law to determine $\angle A$.

$$55^2 = 26^2 + 32^2 - 2(26)(32)\cos A$$

I substituted the values of a, b, and c into the formula.

$$\cos A = \frac{55^2 - (26^2 + 32^2)}{-2(26)(32)}$$

$$\angle A = \cos^{-1}\left(\frac{55^2 - (26^2 + 32^2)}{-2(26)(32)}\right)$$

I used the inverse cosine function on my calculator to determine $\angle A$.

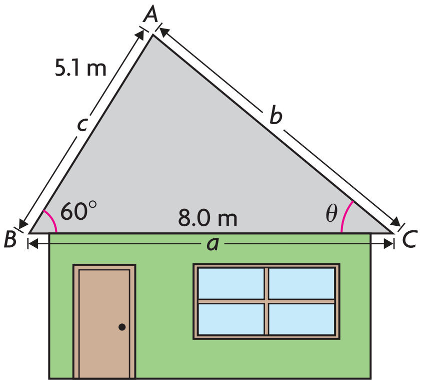
$$\angle A \doteq 143°$$

Given $\triangle ABC$, $\angle A$ is about 143°.

EXAMPLE 3 | Solving a problem by using the cosine and the sine laws

Mitchell wants his 8.0 wide house to be heated with a solar hot-water system. The tubes form an array that is 5.1 m long. In order for the system to be effective, the array must be installed on the south side of the roof and the roof needs to be inclined by 60°. If the north side of the roof is inclined more than 40°, the roof will be too steep for Mitchell to install the system himself. Will Mitchell be able to install this system by himself?

Serina's Solution



I drew a sketch of the situation. I wanted to use the sine law to determine angle θ to solve the problem. But before I could do that, I needed to determine the length of side b.

$$b^2 = a^2 + c^2 - 2ac\cos B$$

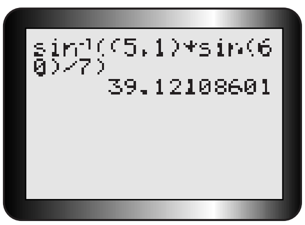
Since I knew two sides and the angle between them, I couldn't use the sine law to determine b. So I used the cosine law.

$$b^2 = (8.0)^2 + (5.1)^2 - 2(8.0)(5.1)\cos 60°$$
$$b^2 = 49.21 \text{ m}^2$$
$$b = \sqrt{49.21}$$
$$b \doteq 7.0 \text{ m}$$

I substituted the values of a, c, and $\angle B$ into the formula. I calculated b by evaluating the right side of the equation and determining its square root.

$$\frac{\sin C}{c} = \frac{\sin B}{b}$$
$$\frac{\sin \theta}{5.1} = \frac{\sin 60°}{7.0}$$
$$\overset{1}{\cancel{5.1}} \times \frac{\sin \theta}{\underset{1}{\cancel{5.1}}} = 5.1 \times \frac{\sin 60°}{7.0}$$
$$\theta = 5.1 \times \frac{\sin 60°}{7.0}$$

I determined $\angle C$ (θ) by using the sine law. Since I needed to solve for an angle, I wrote the sine law with the angles in the numerators. I multiplied both sides of the equation by 5.1 to solve for sin θ.



I used the inverse sine function on my calculator to determine angle θ.

$$\theta \doteq 39°$$

Since Mitchell's roof is inclined about 39° on the north side, he will be able to install the solar hot-water system by himself.

In Summary

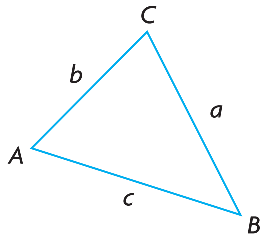
Key Idea

- Given any triangle, the cosine law can be used if you know
  - two sides and the angle contained between those sides (SAS) or
  - all three sides (SSS)

Need to Know

- The cosine law states that in any $\triangle ABC$,

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$



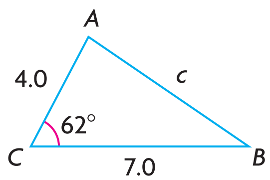
- If $\angle A = 90°$ and $\angle A$ is the contained angle, then the cosine law simplifies to the Pythagorean theorem:

$$a^2 = b^2 + c^2 - 2bc\cos 90°$$
$$a^2 = b^2 + c^2 - 2bc(0)$$
$$a^2 = b^2 + c^2$$

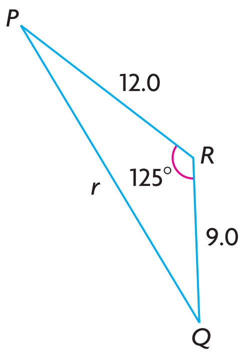
CHECK Your Understanding

1. Determine each unknown side length to the nearest tenth.

a)

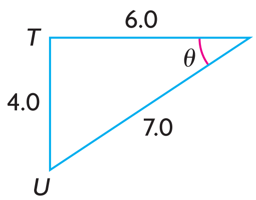


b)



2. For each triangle, determine the value of θ to the nearest degree.

a)



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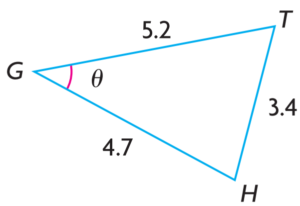
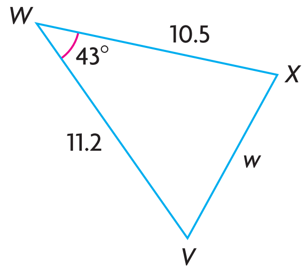
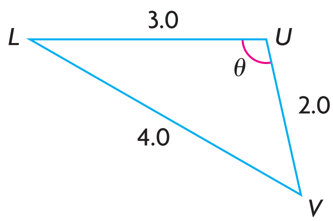


PRACTISING

3. a) Determine w to the nearest tenth.



b) Determine the value of θ to the nearest degree.

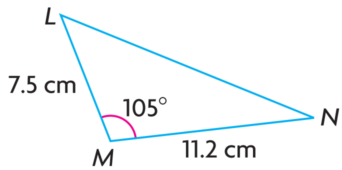


c) In $\triangle ABC$, $a = 11.5$, $b = 8.3$, and $c = 6.6$. Calculate $\angle A$ to the nearest degree.

d) In $\triangle PQR$, $q = 25.1$, $r = 71.3$, and $\cos P = \frac{1}{4}$. Calculate p to the nearest tenth.

4. **K** Calculate each unknown angle to the nearest degree and each unknown length to the nearest tenth of a centimetre.

a)



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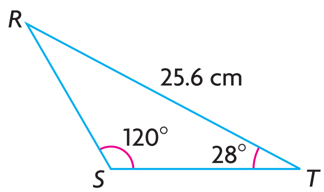
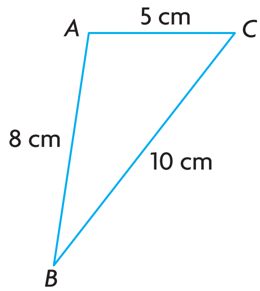
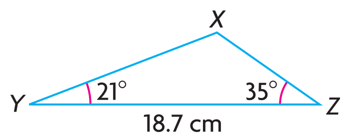
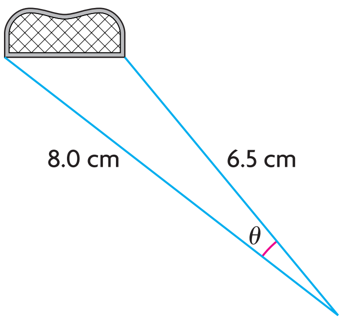


c)

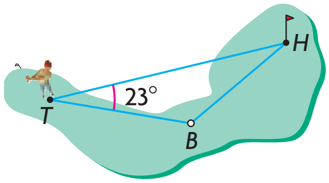


d)

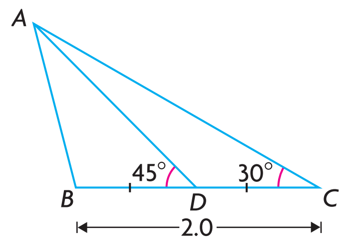




5. **A** The posts of a hockey goal are 2.0 m apart. A player attempts to score by shooting the puck along the ice from a point 6.5 m from one post and 8.0 m from the other. Within what angle θ must the shot be made? Round your answer to the nearest degree.



6. While golfing, Sahar hits a tee shot from T toward a hole at H, but the ball veers 23° and lands at B. The scorecard says that H is 270 m from T. If Sahar walks 160 m to the ball (B), how far, to the nearest metre, is the ball from the hole?

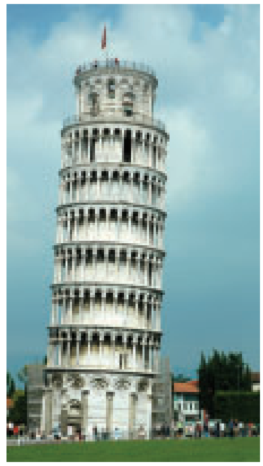


7. Given $\triangle ABC$ at the right, $BC = 2.0$ and D is the midpoint of BC. Determine AB, to the nearest tenth, if $\angle ADB = 45°$ and $\angle ACB = 30°$.

8. Two forest fire towers, A and B, are 20.3 km apart. From tower A, the bearing of tower B is 70°. The ranger in each tower observes a fire and radios the bearing of the fire from the tower. The bearing from tower A is 25° and from tower B is 345°. How far, to the nearest tenth of a kilometre, is the fire from each tower?

9. **T** Two roads intersect at an angle of 15°. Darryl is standing on one of the roads 270 m from the intersection.
   - **a)** Create a question that requires using the sine law to solve it. Include a complete solution and a sketch.
   - **b)** Create a question that requires using the cosine law to solve it. Include a complete solution and a sketch.

10. The Leaning Tower of Pisa is 55.9 m tall and leans 5.5° from the vertical. If its shadow is 90.0 m long, what is the distance from the top of the tower to the top edge of its shadow? Assume that the ground around the tower is level. Round your answer to the nearest metre.



11. **C** The side lengths and the interior angles of any triangle can be determined by using the cosine law, the sine law, or a combination of both. Sketch a triangle and state the minimum information required to use
   - **a)** the cosine law
   - **b)** both laws

   Under each sketch, use the algebraic representation of the law to show how to determine all unknown quantities.

Extending

12. The interior angles of a triangle are 120°, 40°, and 20°. The longest side is 10 cm longer than the shortest side. Determine the perimeter of the triangle to the nearest centimetre.

13. For each situation, determine all unknown side lengths to the nearest tenth of a centimetre and/or all unknown interior angles to the nearest degree. If more than one solution is possible, state all possible answers.
   - **a)** A triangle has exactly one angle measuring 45° and sides measuring 5.0 cm, 7.4 cm, and 10.0 cm.
   - **b)** An isosceles triangle has at least one interior angle of 70° and at least one side of length 11.5 cm.

14. Two hot-air balloons are moored to level ground below, each at a different location. An observer at each location determines the angle of elevation to the opposite balloon as shown at the right. The observers are 2.0 km apart.
   - **a)** What is the distance separating the balloons, to the nearest tenth of a kilometre?
   - **b)** Determine the difference in height (above the ground) between the two balloons. Round your answer to the nearest metre.