5.5 Trigonometric Identities

GOAL

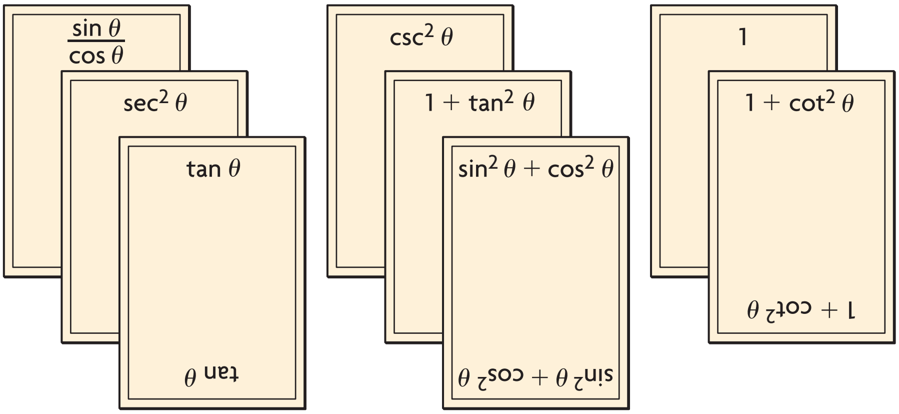
Prove simple trigonometric identities.

LEARN ABOUT the Math

Trident Fish is a game involving a deck of cards, each of which has a mathematical expression written on it. The object of the game is to lay down pairs of equivalent expressions so that each pair forms an **identity**. Suppose you have the cards shown.

identity

a mathematical statement that is true for all values of the given variables. If the identity involves fractions, the denominators cannot be zero. Any restrictions on a variable must be stated.



? What identities can you form with these cards?

EXAMPLE 1 Proving the quotient identity by rewriting in terms of *x*, *y*, and *r*

Prove the quotient identity $\tan \theta = \frac{\sin \theta}{\cos \theta}$ for all angles θ, where $0° \leq \theta \leq 360°$.

Jinji's Solution

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\text{L.S.} = \tan \theta \qquad \text{R.S.} = \frac{\sin \theta}{\cos \theta}$$

I separated the left and the right sides so that I could show that both expressions are equivalent.

$$\text{L.S.} = \frac{y}{x} \qquad \text{R.S.} = \frac{\left(\frac{y}{r}\right)}{\left(\frac{x}{r}\right)}$$

I wrote sin θ, tan θ, and cos θ in terms of x, y, and r, since θ can be greater than 90°.

$$= \frac{y}{\cancel{r}_1} \times \frac{\cancel{r}^1}{x}$$

I simplified the right side by multiplying the numerator by the reciprocal of the denominator.

$$= \frac{y}{x}$$

$$= \text{L.S.}$$

Since the left side works out to the same expression as the right side, the original equation is an identity.

$\therefore \tan\theta = \frac{\sin\theta}{\cos\theta}$ for all angles θ, where $0° \leq \theta \leq 360°$ and $\theta \neq 90°$ or $270°$.

Tan θ is undefined when $\cos\theta = 0$. This occurs when $\theta = 90°$ or $270°$. So θ cannot equal these two values.

EXAMPLE 2 Proving the Pythagorean identity by rewriting in terms of *x*, *y*, and *r*

Prove the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ for all angles θ, where $0° \leq \theta \leq 360°$.

Lisa's Solution

$$\sin^2\theta + \cos^2\theta = 1$$

$$\text{L.S.} = \sin^2\theta + \cos^2\theta \qquad \text{R.S.} = 1$$

I separated the left and the right sides so that I could show that both expressions are equivalent.

$$= \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2$$

I wrote sin θ and cos θ in terms of x, y, and r, since θ can be greater than 90°. Then I simplified.

$$= \frac{y^2}{r^2} + \frac{x^2}{r^2}$$

$$= \frac{y^2 + x^2}{r^2}$$

I knew that $r^2 = x^2 + y^2$ from the Pythagorean theorem. I used this equation to further simplify the left side.

$$= \frac{r^2}{r^2}$$

$$= 1$$

$$= \text{R.S.}$$

Since the left side works out to the same expression as the right side, the original equation is an identity.

$\therefore \sin^2\theta + \cos^2\theta = 1$ for all angles θ, where $0° \leq \theta \leq 360°$.

EXAMPLE 3 Proving an identity by using a common denominator

Prove that $1 + \cot^2\theta = \csc^2\theta$ for all angles θ between 0° and 360° except 0°, 180°, and 360°.

Pedro's Solution

$1 + \cot^2\theta = \csc^2\theta$

L.S. $= 1 + \cot^2\theta$ R.S. $= \csc^2\theta$

I separated the left and the right sides so that I could show that both expressions are equivalent.

$= 1 + \left(\frac{\cos\theta}{\sin\theta}\right)^2$ $= \left(\frac{1}{\sin\theta}\right)^2$

$= 1 + \frac{\cos^2\theta}{\sin^2\theta}$ $= \frac{1}{\sin^2\theta}$

I expressed the reciprocal trigonometric ratios in terms of the primary ratios $\sin\theta$ and $\cos\theta$. I knew that $\cot\theta = \frac{1}{\tan\theta}$ and $\tan\theta = \frac{\sin\theta}{\cos\theta}$, so $\cot\theta = \frac{\cos\theta}{\sin\theta}$. Since θ can't be 0°, 180°, or 360°, $\sin\theta \neq 0$, I don't have any term that is undefined.

$= \frac{\sin^2\theta}{\sin^2\theta} + \frac{\cos^2\theta}{\sin^2\theta}$

On the left side, I expressed 1 as $\frac{\sin^2\theta}{\sin^2\theta}$ to get a common denominator of $\sin^2\theta$.

$= \frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta}$

I used the Pythagorean identity $\sin^2\theta + \cos^2\theta = 1$ to simplify the numerator.

$= \frac{1}{\sin^2\theta}$

$=$ R.S.

Since the left side works out to the same expression as the right side, the original equation is an identity.

$\therefore 1 + \cot^2\theta = \csc^2\theta$ for all angles θ between 0° and 360° except 0°, 180°, and 360°.

Reflecting

A. What strategy would you use to prove the identity $1 + \tan^2 \theta = \sec^2 \theta$? What restrictions does θ have?

B. When is it important to consider restrictions on θ?

C. How might you create new identities based on Examples 1 and 2?

APPLY the Math

EXAMPLE 4 Proving an identity by factoring

Prove that $\tan \phi = \frac{\sin \phi + \sin^2 \phi}{(\cos \phi)(1 + \sin \phi)}$ for all angles ϕ between 0° and 360°, where $\cos \phi \neq 0$.

Jamal's Solution

$$\tan \phi = \frac{\sin \phi + \sin^2 \phi}{(\cos \phi)(1 + \sin \phi)}$$

I separated the left and the right sides so that I could show that both expressions are equivalent.

$$\text{L.S.} = \tan \phi \qquad \text{R.S.} = \frac{\sin \phi + \sin^2 \phi}{(\cos \phi)(1 + \sin \phi)}$$

$$= \frac{\sin \phi}{\cos \phi} \qquad = \frac{\sin \phi \overset{1}{\cancel{(1 + \sin \phi)}}}{(\cos \phi)\underset{1}{\cancel{(1 + \sin \phi)}}}$$

$$= \frac{\sin \phi}{\cos \phi}$$

I knew that $\tan \phi$ could be written as $\frac{\sin \phi}{\cos \phi}$. The right side is more complicated, so I factored out $\sin \phi$ from the numerator. Since $\cos \phi \neq 0$, the denominator will not be 0. I divided the numerator and denominator by the factor $1 + \sin \phi$.

$$= \text{L.S.}$$

$\therefore \tan \phi = \frac{\sin \phi + \sin^2 \phi}{(\cos \phi)(1 + \sin \phi)}$ for all angles ϕ between 0° and 360°, where $\cos \phi \neq 0$.

Since the left side works out to the same expression as the right side, the original equation is an identity.

In Summary

Key Ideas

- A trigonometric identity is an equation involving trigonometric ratios that is true for all values of the variable.
- Some trigonometric identities are a result of a definition, while others are derived from relationships that exist among trigonometric ratios.

Need to Know

- Some trigonometric identities that are important to remember are shown below ($0° \leq \theta \leq 360°$).

Identities Based on Definitions	Identities Derived from Relationships	
Reciprocal Identities	**Quotient Identities**	**Pythagorean Identities**
$\csc \theta = \frac{1}{\sin \theta}$, where $\sin \theta \neq 0$	$\tan \theta = \frac{\sin \theta}{\cos \theta}$, where $\cos \theta \neq 0$	$\sin^2 \theta + \cos^2 \theta = 1$
$\sec \theta = \frac{1}{\cos \theta}$, where $\cos \theta \neq 0$	$\cot \theta = \frac{\cos \theta}{\sin \theta}$, where $\sin \theta \neq 0$	$1 + \tan^2 \theta = \sec^2 \theta$
$\cot \theta = \frac{1}{\tan \theta}$, where $\tan \theta \neq 0$		$1 + \cot^2 \theta = \csc^2 \theta$

- To prove that a given trigonometric equation is an identity, both sides of the equation need to be shown to be equivalent. This can be done by
  - simplifying the more complicated side until it is identical to the other side or manipulating both sides to get the same expression
  - rewriting all trigonometric ratios in terms of x, y, and r
  - rewriting all expressions involving tangent and the reciprocal trigonometric ratios in terms of sine and cosine
  - applying the Pythagorean identity where appropriate
  - using a common denominator or factoring as required

CHECK Your Understanding

1. Prove each identity by writing all trigonometric ratios in terms of x, y, and r. State the restrictions on θ.

   a) $\cot\theta = \dfrac{\cos\theta}{\sin\theta}$

   b) $\tan\theta\cos\theta = \sin\theta$

   c) $\csc\theta = \dfrac{1}{\sin\theta}$

   d) $\cos\theta\sec\theta = 1$

2. Simplify each expression.

   a) $(1 - \sin\alpha)(1 + \sin\alpha)$

   b) $\dfrac{\tan\alpha}{\sin\alpha}$

   c) $\cos^2\alpha + \sin^2\alpha$

   d) $\cot\alpha\sin\alpha$

3. Factor each expression.

   a) $1 - \cos^2\theta$

   b) $\sin^2\theta - \cos^2\theta$

   c) $\sin^2\theta - 2\sin\theta + 1$

   d) $\cos\theta - \cos^2\theta$

PRACTISING

4. Prove that $\frac{\cos^2\phi}{1 - \sin\phi} = 1 + \sin\phi$, where $\sin\phi \neq 1$, by expressing $\cos^2\phi$ in terms of $\sin\phi$.

5. Prove each identity. State any restrictions on the variables.

   a) $\dfrac{\sin x}{\tan x} = \cos x$

   b) $\dfrac{\tan\theta}{\cos\theta} = \dfrac{\sin\theta}{1 - \sin^2\theta}$

   c) $\dfrac{1}{\cos\alpha} + \tan\alpha = \dfrac{1 + \sin\alpha}{\cos\alpha}$

   d) $1 - \cos^2\theta = \sin\theta\cos\theta\tan\theta$

6. Mark claimed that $\frac{1}{\cot\theta} = \tan\theta$ is an identity. Marcia let $\theta = 30°$ and found that both sides of the equation worked out to $\frac{1}{\sqrt{3}}$. She said that this proves that the equation is an identity. Is Marcia's reasoning correct? Explain.

7. Simplify each trigonometric expression.

   a) $\sin\theta\cot\theta - \sin\theta\cos\theta$

   b) $\cos\theta(1 + \sec\theta)(\cos\theta - 1)$

   c) $(\sin x + \cos x)(\sin x - \cos x) + 2\cos^2 x$

   d) $\dfrac{\csc^2\theta - 3\csc\theta + 2}{\csc^2\theta - 1}$

8. Prove each identity. State any restrictions on the variables.

   a) $\dfrac{\sin^2\phi}{1 - \cos\phi} = 1 + \cos\phi$

   b) $\dfrac{\tan^2\alpha}{1 + \tan^2\alpha} = \sin^2\alpha$

   c) $\cos^2 x = (1 - \sin x)(1 + \sin x)$

   d) $\sin^2\theta + 2\cos^2\theta - 1 = \cos^2\theta$

   e) $\sin^4\alpha - \cos^4\alpha = \sin^2\alpha - \cos^2\alpha$

   f) $\tan\theta + \dfrac{1}{\tan\theta} = \dfrac{1}{\sin\theta\cos\theta}$

9. **A** Farah claims that if you separate both sides of an equation into two functions and graph them on the same *xy*-axes on a graphing calculator, you can use the result to prove that the equation is an identity.

a) Is her claim correct? Justify your answer.

b) Discuss the limitations of her approach.

10. Is $\csc^2 \theta + \sec^2 \theta = 1$ an identity? Prove that it is true or demonstrate why it is false.

11. **K** Prove that $\sin^2 x\left(1 + \frac{1}{\tan^2 x}\right) = 1$, where $\sin x \neq 0$.

12. **T** Prove each identity. State any restrictions on the variables.

a) $\dfrac{\sin^2 \theta + 2\cos \theta - 1}{\sin^2 \theta + 3\cos \theta - 3} = \dfrac{\cos^2 \theta + \cos \theta}{-\sin^2 \theta}$

b) $\sin^2 \alpha - \cos^2 \alpha - \tan^2 \alpha = \dfrac{2\sin^2 \alpha - 2\sin^4 \alpha - 1}{1 - \sin^2 \alpha}$

13. **C** Show how you can create several new identities from the identity $\sin^2 \theta + \cos^2 \theta = 1$ by adding, subtracting, multiplying, or dividing both sides of the equation by the same expression.

Extending

14. **a)** Which equations are not identities? Justify your answers.

b) For those equations that are identities, state any restrictions on the variables.

i) $(1 - \cos^2 x)(1 - \tan^2 x) = \dfrac{\sin^2 x - 2\sin^4 x}{1 - \sin^2 x}$

ii) $1 - 2\cos^2 \phi = \sin^4 \phi - \cos^4 \phi$

iii) $\dfrac{\sin \theta \tan \theta}{\sin \theta + \tan \theta} = \sin \theta \tan \theta$

iv) $\dfrac{1 + 2\sin \beta \cos \beta}{\sin \beta + \cos \beta} = \sin \beta + \cos \beta$

v) $\dfrac{1 - \cos \beta}{\sin \beta} = \dfrac{\sin \beta}{1 + \cos \beta}$

vi) $\dfrac{\sin x}{1 + \cos x} = \csc x - \cot x$