5.2 Evaluating Trigonometric Ratios for Special Angles

GOAL

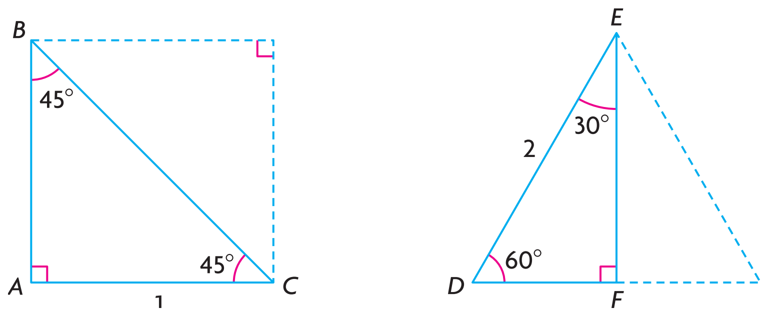
Evaluate exact values of sine, cosine, and tangent for specific angles.

YOU WILL NEED

- ruler
- protractor

LEARN ABOUT the Math

The diagonal of a square of side length 1 unit creates two congruent right isosceles triangles. The height of an equilateral triangle of side length 2 units creates two congruent right scalene triangles.

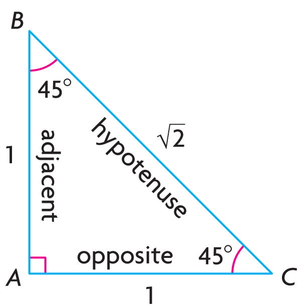


? How can isosceles $\triangle ABC$ and scalene $\triangle DEF$ be used to determine the exact values of the primary trigonometric ratios for 30°, 45°, and 60° angles?

EXAMPLE 1 Evaluating exact values of the trigonometric ratios for a 45° angle

Use $\triangle ABC$ to calculate exact values of sine, cosine, and tangent for 45°.

Carol's Solution



$$BC^2 = AB^2 + AC^2$$
$$BC^2 = 1^2 + 1^2$$
$$BC^2 = 2$$
$$BC = \sqrt{2}$$

I labelled the sides of the triangle relative to $\angle B$. The triangle is isosceles with equal sides of length 1. I used the Pythagorean theorem to calculate the length of the hypotenuse.

$$\sin B = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos B = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan B = \frac{\text{opposite}}{\text{adjacent}}$$

I wrote the primary trigonometric ratios for $\angle B$.

$$\sin 45° = \frac{1}{\sqrt{2}} \qquad \cos 45° = \frac{1}{\sqrt{2}} \qquad \tan 45° = \frac{1}{1}$$

$$= \frac{1 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} \qquad = \frac{1 \times \sqrt{2}}{\sqrt{2} \times \sqrt{2}} \qquad = 1$$

If I multiplied both the numerator and denominator by $\sqrt{2}$, I would get an equivalent number with a whole-number denominator.

$$= \frac{\sqrt{2}}{2} \qquad = \frac{\sqrt{2}}{2}$$

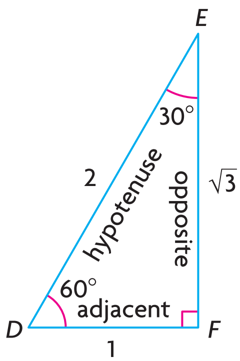
This would be an easier number to use to estimate the size, since I knew that $\sqrt{2}$ is about 1.4, so half of it is about 0.7.

The exact values of sine, cosine, and tangent for 45° are $\frac{\sqrt{2}}{2}$, $\frac{\sqrt{2}}{2}$, and 1, respectively.

EXAMPLE 2 Evaluating exact values of the trigonometric ratios for 30° and 60° angles

Use $\triangle DEF$ to calculate exact values of sine, cosine, and tangent for 30° and 60°.

Trevor's Solution



$$DE^2 = DF^2 + EF^2$$
$$2^2 = 1^2 + EF^2$$
$$4 = 1 + EF^2$$
$$3 = EF^2$$
$$\sqrt{3} = EF$$

I labelled the sides of the triangle relative to $\angle D$. Since the height of an equilateral triangle divides the triangle into two smaller identical triangles, DF is equal to $\frac{1}{2}DE$. So DF must be 1. I used the Pythagorean theorem to calculate the length of EF.

$$\sin D = \frac{\text{opposite}}{\text{hypotenuse}} \qquad \cos D = \frac{\text{adjacent}}{\text{hypotenuse}} \qquad \tan D = \frac{\text{opposite}}{\text{adjacent}}$$

I wrote the primary trigonometric ratios for $\angle D$.

$$\sin D = \frac{EF}{DE} \qquad \cos D = \frac{DF}{DE} \qquad \tan D = \frac{EF}{DF}$$

$$\sin 60° = \frac{\sqrt{3}}{2} \qquad \cos 60° = \frac{1}{2} \qquad \tan 60° = \frac{\sqrt{3}}{1}$$

$$= \sqrt{3}$$

$$\sin E = \frac{DF}{DE} \qquad \cos E = \frac{EF}{DE} \qquad \tan E = \frac{DF}{EF}$$

I wrote the primary trigonometric ratios for $\angle E$ in terms of the sides of the triangle.

$\sin 30° = \frac{1}{2}$ $\qquad \cos 30° = \frac{\sqrt{3}}{2}$ $\qquad \tan 30° = \frac{1}{\sqrt{3}}$

$$= \frac{1 \times \sqrt{3}}{\sqrt{3} \times \sqrt{3}}$$

$$= \frac{\sqrt{3}}{3}$$

If I multiplied both the numerator and denominator by $\sqrt{3}$, I would get an equivalent number with a whole-number denominator. This is an easier number to estimate, since $\sqrt{3}$ is about 1.7, so a third of it is about 0.57.

$\sin E = \cos D \qquad \cos E = \sin D \qquad \tan E = \cot D$

$\sin 30° = \cos 60° \qquad \cos 30° = \sin 60° \qquad \tan 30° = \cot 60°$

I noticed that sin *E* and cos *E* are equal to cos *D* and sin *D*, respectively. I also noticed that tan *E* is equal to the reciprocal of tan *D*.

The exact values of sine, cosine, and tangent for 30° are $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$, and $\frac{\sqrt{3}}{3}$, respectively and for 60° are $\frac{\sqrt{3}}{2}$, $\frac{1}{2}$, and $\sqrt{3}$, respectively.

Reflecting

A. In Example 1, would you get the same results if you used $\angle C$ for the 45° angle instead of $\angle B$? Explain.

B. Explain how sin 30° and cos 60° are related.

C. In Example 2, explain why the reciprocal ratios of tan 30° and cot 60° are equal.

D. How can remembering that a 30°–60°–90° triangle is half of an equilateral triangle and that a 45°–45°–90° triangle is isosceles help you recall the exact values of the primary trigonometric ratios for the angles in those triangles?

APPLY the Math

EXAMPLE 3 Determining the exact value of a trigonometric expression

Determine the exact value of $(\sin 45°)(\cos 45°) + (\sin 30°)(\sin 60°)$.

Tina's Solution

$(\sin 45°)(\cos 45°) + (\sin 30°)(\sin 60°)$

$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$

I substituted the exact values of each trigonometric ratio.

$$= \frac{2}{4} + \frac{\sqrt{3}}{4}$$

I evaluated the expression by multiplying, then adding the numerators.

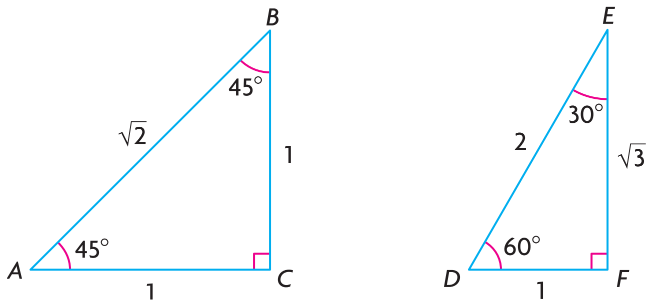
$$= \frac{2 + \sqrt{3}}{4}$$

The exact value is $\frac{2 + \sqrt{3}}{4}$.

In Summary

Key Idea

- The exact values of the primary trigonometric ratios for 30°, 45°, and 60° angles can be found by using the appropriate ratios of sides in isosceles right triangles and half-equilateral triangles with right angles. These are often referred to as "special triangles."



θ	sin θ	cos θ	tan θ
30°	$\frac{1}{2} = 0.5$	$\frac{\sqrt{3}}{2} \doteq 0.8660$	$\frac{\sqrt{3}}{3} \doteq 0.5774$
45°	$\frac{\sqrt{2}}{2} \doteq 0.7071$	$\frac{\sqrt{2}}{2} \doteq 0.7071$	1
60°	$\frac{\sqrt{3}}{2} \doteq 0.8660$	$\frac{1}{2} = 0.5$	$\sqrt{3} \doteq 1.7321$

Need to Know

- Since tan 45° = 1, angles between 0° and 45° have tangent ratios that are less than 1, and angles between 45° and 90° have tangent ratios greater than 1.
- If a right triangle has one side that is half the length of the hypotenuse, the angle opposite that one side is always 30°.
- If a right triangle has two equal sides, then the angles opposite those sides are always 45°.

CHECK Your Understanding

1. **a)** Draw a right triangle that has one angle measuring 30°. Label the sides using the lengths $\sqrt{3}$, 2, and 1. Explain your reasoning.
b) Identify the adjacent and opposite sides relative to the 30° angle.
c) Identify the adjacent and opposite sides relative to the 60° angle.

2. **a)** Draw a right triangle that has one angle measuring 45°. Label the sides using the lengths 1, 1, and $\sqrt{2}$. Explain your reasoning.
b) Identify the adjacent and opposite sides relative to one of the 45° angles.

3. State the exact values.
a) sin 60° **b)** cos 30° **c)** tan 45° **d)** cos 45°

PRACTISING

4. Determine the exact value of each trigonometric expression.
K
a) $\sin 30° \times \tan 60° - \cos 30°$
b) $2 \cos 45° \times \sin 45°$
c) $\tan^2 30° - \cos^2 45°$
d) $1 - \dfrac{\sin 45°}{\cos 45°}$

Communication *Tip*

$\tan^2 30° = (\tan 30°)(\tan 30°)$. The expression is squared, not the angle.

5. Using exact values, show that $\sin^2 \theta + \cos^2 \theta = 1$ for each angle.
a) $\theta = 30°$ **b)** $\theta = 45°$ **c)** $\theta = 60°$

6. Using exact values, show that $\dfrac{\sin \theta}{\cos \theta} = \tan \theta$ for each angle.
a) $\theta = 30°$ **b)** $\theta = 45°$ **c)** $\theta = 60°$

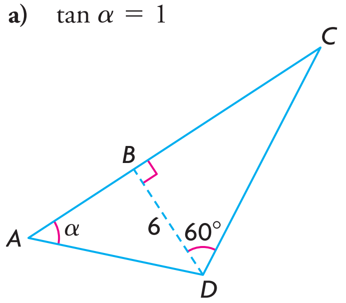
7. Using the appropriate special triangle, determine θ if $0° \leq \theta \leq 90°$.
a) $\sin \theta = \dfrac{\sqrt{3}}{2}$
b) $\sqrt{3} \tan \theta = 1$
c) $2\sqrt{2} \cos \theta = 2$
d) $2 \cos \theta = \sqrt{3}$

8. A 5 m stepladder propped against a classroom wall forms an angle of 30° with the wall. Exactly how far is the top of the ladder from the floor? Express your answer in radical form. What assumption did you make?
A

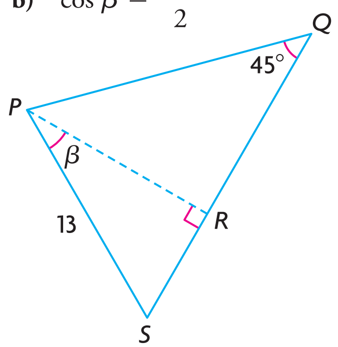
9. Show that $\tan 30° + \dfrac{1}{\tan 30°} = \dfrac{1}{\sin 30° \cos 30°}$.

10. A baseball diamond forms a square of side length 27.4 m. Sarah says that she used a special triangle to calculate the distance between home plate and second base.
a) Describe how Sarah might calculate this distance.
b) Use Sarah's method to calculate this distance to the nearest tenth of a metre.

11. Determine the exact area of each large triangle.
T
a) $\tan \alpha = 1$



b) $\cos \beta = \dfrac{\sqrt{3}}{2}$



12. To claim a prize in a contest, the following skill-testing question was asked: Calculate $\sin 45°(1 - \cos 30°) + 5 \tan 60°(\sin 60° - \tan 30°)$.
C
a) Louise used a calculator to evaluate the expression. Determine her answer to three decimal places.
b) Megan used exact values. Determine her answer in radical form.
c) Only Megan received the prize. Explain why this might have occurred.

Extending

13. If $\cot \alpha = \sqrt{3}$, calculate $(\sin \alpha)(\cot \alpha) - \cos^2 \alpha$ exactly.

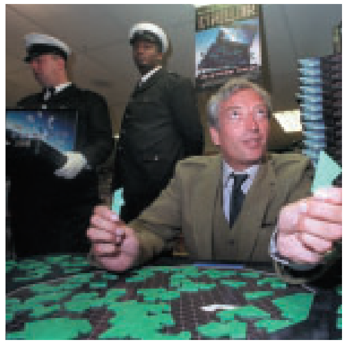
14. If $\csc \beta = 2$, calculate $\frac{\tan \beta}{\sec \beta} - \sin^2 \beta$ exactly.

15. Using exact values, show that $1 + \cot^2 \theta = \csc^2 \theta$ for each angle.
a) $\theta = 30°$ **b)** $\theta = 45°$ **c)** $\theta = 60°$

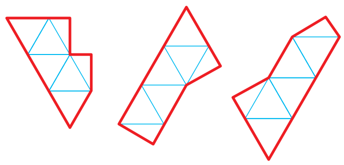
Curious *Math*

The Eternity Puzzle

Eternity, a puzzle created by Christopher Monckton, consists of 209 different pieces. Each piece is made up of twelve $30°-60°-90°$ triangles. The puzzle was introduced in Britain in June 1999, and the goal was to arrange the pieces into the shape of a dodecagon (12-sided polygon). Monckton provided six clues to solve his puzzle, and a £1 000 000 award (about \$2 260 000 Canadian dollars) was offered for the first solution. It turned out that the puzzle didn't take an eternity to solve after all! Alex Selby and Oliver Riordan presented their solution on May 15, 2000, and collected the prize.



A second solution was found by Guenter Stertenbrink shortly afterwards. Interestingly, all three mathematicians ignored Monckton's clues and found their own answers. Monckton's solution remains unknown.

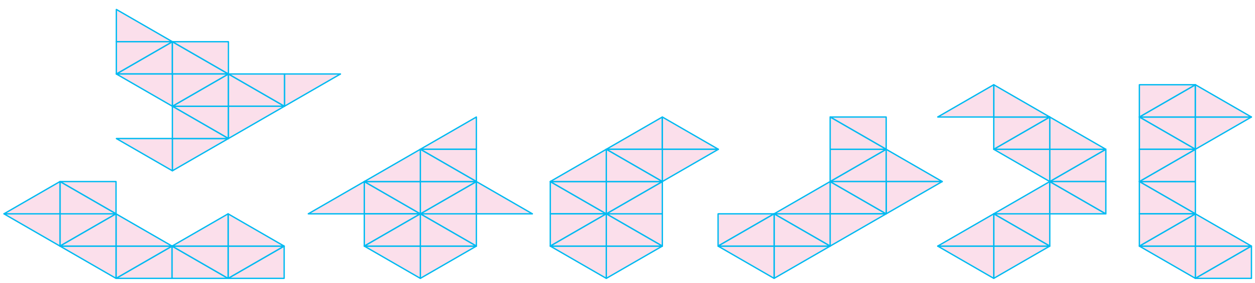


1. Consider the first three pieces of the Eternity puzzle. Each contains twelve $30°-60°-90°$ triangles. Suppose one such triangle has side lengths of 1, $\sqrt{3}$, and 2, respectively.

a) For each puzzle piece, determine the perimeter. Write your answer in radical form.

b) Calculate the area of each puzzle piece. Round your answer to the nearest tenth of a square unit.

2. The seven puzzle pieces shown can be fit together to form a convex shape. Copy these pieces and see if you can find a solution.



Extending

13. If $\cot \alpha = \sqrt{3}$, calculate $(\sin \alpha)(\cot \alpha) - \cos^2 \alpha$ exactly.

14. If $\csc \beta = 2$, calculate $\dfrac{\tan \beta}{\sec \beta} - \sin^2 \beta$ exactly.

15. Using exact values, show that $1 + \cot^2 \theta = \csc^2 \theta$ for each angle.
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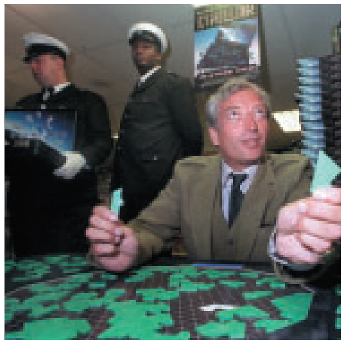

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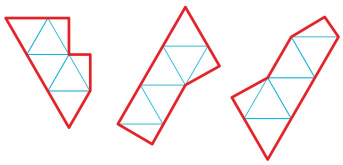

piece 1 piece 2 piece 3

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