4.7 Applications Involving Exponential Functions

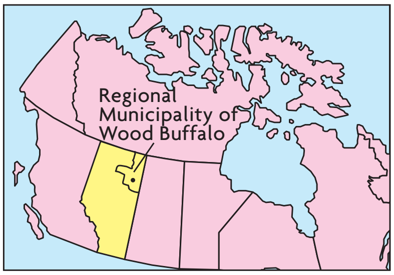
YOU WILL NEED

- graphing calculator

GOAL

Use exponential functions to solve problems involving exponential growth and decay.

The regional municipality of Wood Buffalo, Alberta, has experienced a large population increase in recent years due to the discovery of one of the world's largest oil deposits. Its population, 35 000 in 1996, has grown at an annual rate of approximately 8%.



? How long will it take for the population to double at this growth rate?

LEARN ABOUT the Math

EXAMPLE 1 Selecting a strategy to determine the doubling rate

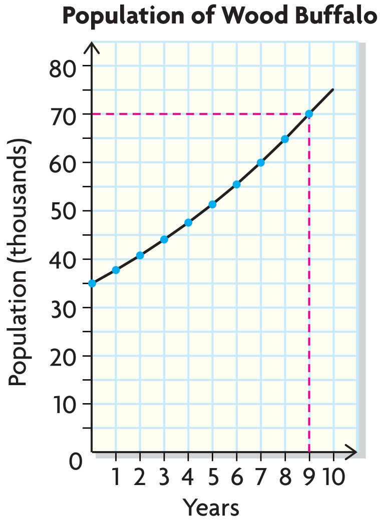
Carter's Solution: Using a Table of Values and a Graph

$0.08(35) + 35 = 35(0.08 + 1) = 35(1.08)$

When you add 8% of a number to the number, the new value is 108% of the old one. This is the same as multiplying by 1.08, so I created the table of values by repeatedly multiplying by 1.08.

I did this 10 times, once for each year, and saw that the population doubled to 70 000 after 9 years of growth.

Time (year from 1996)	0	1	2	3	4	5	6	7	8	9	10
Population (thousands)	35.0	37.8	40.8	44.1	47.6	51.4	55.5	60.0	64.8	70.0	75.6



I plotted the points and drew a smooth curve through the data.

I drew a horizontal line across the graph at 70 000 and saw that it touched the curve at 9 years.

Sonja's Solution: Creating an Algebraic Model

$P(1) = 35(1.08) = 37.8$

$P(2) = 37.8(1.08) = 40.8$

Substituting P(1) into P(2):

$P(2) = 35(1.08)(1.08)$

$= 35(1.08)^2$

So, $P(3) = 35(1.08)^2(1.08)$

$= 35(1.08)^3$

To calculate the population after 1 year, I needed to multiply 35 by 1.08. For each additional year, I repeatedly multiply by 1.08. Repeated multiplication can be represented with exponents. The value of the exponent will correspond to the number for the year.

This led to an algebraic model.

Therefore, $P(n) = 35(1.08)^n$

$P(6) = 35(1.08)^6$

$= 35(1.586\ 874\ 323)$

$= 55.540\ 601\ 3$

$P(9) = 35(1.08)^9$

$= 35(1.999\ 004\ 627)$

$= 69.965\ 161\ 95 \doteq 70$

The population would double in approximately 9 years at an 8% rate of growth.

Since population is a function of time, I expressed the relationship in function notation. I used $P(n)$, where the exponent, n, would represent the number of years after 1996 and $P(n)$ would represent the population in thousands.

I guessed that it would take 6 years for the population to double. I substituted $n = 6$ into the expression for the function, but it was too low.

I tried values for n until I got an answer that was close to the target of 70; $n = 9$ was pretty close.

Reflecting

A. Which features of the function indicate that it is exponential?

B. Describe what each part of the equation $P(n) = 35(1.08)^n$ represents in the context of the problem *and* the features of the graph.

C. Compare Carter's and Sonja's solutions. Which approach do you think is better? Why?

APPLY the Math

EXAMPLE 2 | Solving an exponential decay problem, given the equation

A 200 g sample of radioactive polonium-210 has a *half-life* of 138 days. This means that every 138 days, the amount of polonium left in a sample is half of the original amount. The mass of polonium, in grams, that remains after t days can be modelled by $M(t) = 200\left(\frac{1}{2}\right)^{\frac{t}{138}}$.

a) Determine the mass that remains after 5 years.

b) How long does it take for this 200 g sample to decay to 110 g?

Zubin's Solution: Using the Algebraic Model

a) $5 \text{ years} = 5(365) \text{ days}$

$= 1825 \text{ days}$

$M(1825) = 200\left(\frac{1}{2}\right)^{\frac{1825}{138}}$

$\doteq 200(0.000\,104\,5)$

$\doteq 0.021$

Since the half-life is measured in days, I converted the number of years to days before substituting into the function.

I used my calculator to determine the answer.

There is approximately 0.02 g of polonium-210 left after 5 years.

b) $M(t) = 200\left(\frac{1}{2}\right)^{\frac{t}{138}}$

I began by writing the equation and substituting the amount of the sample remaining.

$110 = 200\left(\frac{1}{2}\right)^{\frac{t}{138}}$

I needed to isolate t in the equation, so I divided each side by 200.

$\frac{110}{200} = \left(\frac{1}{2}\right)^{\frac{t}{138}}$

I didn't know how to isolate t, so I used guess and check to find the answer.

$$0.55 = \left(\frac{1}{2}\right)^{\frac{t}{138}}$$

$$M(100) = 200\left(\frac{1}{2}\right)^{\frac{100}{138}}$$

$$\doteq 121 \text{ g}$$

I knew that if the exponent was 1 (t = 138 days), the original amount would be halved, but the amount I needed to find was 110 g, so the exponent needed to be less than 1. I guessed 100 days, which I substituted into the original equation. I calculated the answer.

It was too high, which meant that my guess was too low.

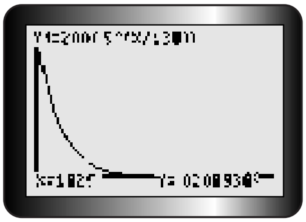
$$M(119) = 200\left(\frac{1}{2}\right)^{\frac{119}{138}}$$

$$\doteq 110 \text{ g}$$

I guessed and checked a few more times until I found the answer of approximately 119 days.

Barry's Solution: Using a Graphical Model

a)



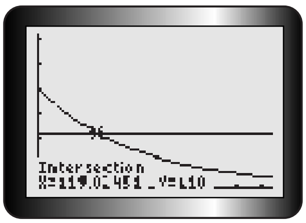
I graphed $M(t)$, then used the value operation. I had to change 5 years into days.

There is about 0.02 g remaining after 5 years.

b)

I graphed $M(t)$, and I graphed a horizontal line to represent 110 g.

I knew that the point where the line met the curve would represent the answer.



I used the "Intersect" operation on the graphing calculator to find the point. The x-value represents the number of days.

It takes approximately 119 days for the sample to decay to 110 g.

Tech Support

For help determining the point(s) of intersection between two functions, see Technical Appendix, B-12.

EXAMPLE 3 Solving a problem by determining an equation for a curve of good fit

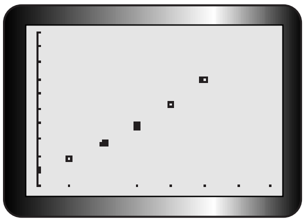
A biologist tracks the population of a new species of frog over several years. From the table of values, determine an equation that models the frog's population growth, and determine the number of years before the population triples.

Year	0	1	2	3	4	5
Population	400	480	576	691	829	995

Tech Support

For help creating scatter plots on a graphing calculator, see Technical Appendix, B-11.

Fred's Solution

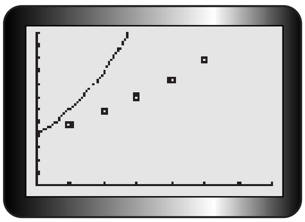


I used my graphing calculator to create a scatter plot.

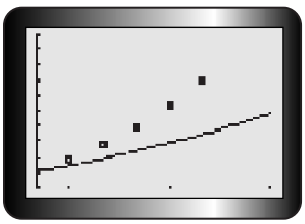
The equation is of the form $P(t) = ab^t$, where

- $P(t)$ represents the population in year t
- a is the initial population
- b is the base of the exponential function

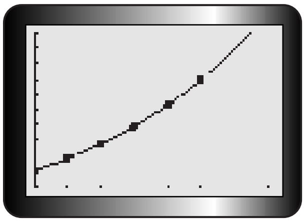
Since the function is increasing, $b > 1$. The initial population occurs when $x = 0$. That means that $a = 400$. If $b = 2$, then the population would have doubled, but it went up by only 80 in the first year, so the value of b must be less than 2.



I tried $b = 1.5$ and entered the equation $P(t) = 400(1.5)^t$ into the equation editor. The graph rose too quickly, so 1.5 is too great for b.



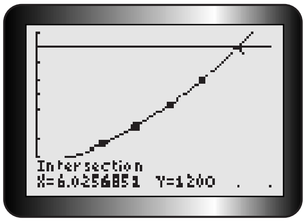
I changed the equation to $P(t) = 400(1.1)^t$. I graphed the equation on the calculator. I checked to see if the curve looked right. It rose too slowly, so b must be between 1.1 and 1.5.



I continued this process until I found a good fit when $b = 1.2$.

The equation that models this population is

$P(t) = 400(1.2)^t$



To determine the year that the population tripled, I graphed the line $P = 1200$ and found the intersection point of the curve and the line.

From this graph, I determined that the frog population tripled in approximately 6 years.

EXAMPLE 4 Representing a real-world problem with an algebraic model

A new car costs $24 000. It loses 18% of its value each year after it is purchased. This is called *depreciation*. Determine the value of the car after 30 months.

Gregg's Solution

$y = ab^x$

The car's value decreases each year. Another way to think about the car *losing* 18% of its value each year is to say that it *keeps* 82% of its value. To determine its value, I multiplied its value in the previous year by 0.82. The repeated multiplication suggested that this relationship is exponential. That makes sense, since this has to be a decreasing function where $0 < b < 1$.

$$V(n) = 24(0.82)^n$$

I used *V* and *n* to remind me of what they represented.

The base of the exponential function that models the value of the car is 0.82. The initial value is \$24 000, which is the value of *a* and the exponent *n* is measured in years.

$$\begin{aligned} n &= 30 \text{ months} \\ &= 30 \div 12 \text{ years} \\ &= 2.5 \text{ years} \end{aligned}$$

I converted 30 months to years to get my answer.

$$\begin{aligned} V(2.5) &= 24(0.82)^{2.5} \\ &= 24(0.608\ 884\ 097) \\ &\doteq 14.6 \end{aligned}$$

The car is worth about \$14 600 after 30 months.

In Summary

Key Ideas

- The exponential function $f(x) = ab^x$ and its graph can be used as a model to solve problems involving exponential growth and decay. Note that
  - $f(x)$ is the final amount or number
  - a is the initial amount or number
  - for exponential growth, $b = 1 +$ growth rate; for exponential decay, $b = 1 -$ decay rate
  - x is the number of growth or decay periods

Need to Know

- For situations that can be modeled by an exponential function:
  - If the *growth rate* (as a percent) is given, then the base of the power in the equation can be obtained by *adding* the rate, as a decimal, to 1. For example, a growth rate of 8% involves multiplying repeatedly by 1.08.
  - If the *decay rate* (as a percent) is given, then the base of the power in the equation is obtained by *subtracting* the rate, as a decimal, from 1. For example, a decay rate of 8% involves multiplying repeatedly by 0.92.
  - One way to tell the difference between growth and decay is to consider whether the quantity in question (e.g., light intensity, population, dollar value) has increased or decreased.
  - The units for the growth/decay rate and for the number of growth/decay periods must be the same. For example, if light intensity decreases "per metre," then the number of decay periods in the equation is measured in metres, too.

CHECK Your Understanding

1. Solve each exponential equation. Express answers to the nearest hundredth of a unit.

a) $A = 250(1.05)^{10}$

b) $P = 9000\left(\frac{1}{2}\right)^{8}$

c) $500 = N_0(1.25)^{1.25}$

d) $625 = P(0.71)^{9}$

2. Complete the table.

	Function	Exponential Growth or Decay?	Initial Value	Growth or Decay Rate
a)	$V(t) = 20(1.02)^t$			
b)	$P(n) = (0.8)^n$			
c)	$A(x) = 0.5(3)^x$			
d)	$Q(w) = 600\left(\frac{5}{8}\right)^w$			

3. The growth in population of a small town since 1996 is given by the function $P(n) = 1250(1.03)^n$.

a) What is the initial population? Explain how you know.

b) What is the growth rate? Explain how you know.

c) Determine the population in the year 2007.

d) In which year does the population reach 2000 people?

4. A computer loses its value each month after it is purchased. Its value as a function of time, in months, is modelled by $V(m) = 1500(0.95)^m$.

a) What is the initial value of the computer? Explain how you know.

b) What is the rate of depreciation? Explain how you know.

c) Determine the value of the computer after 2 years.

d) In which month after it is purchased does the computer's worth fall below $900?

PRACTISING

5. In 1990, a sum of $1000 is invested at a rate of 6% per year for 15 years.

a) What is the growth rate?

b) What is the initial amount?

c) How many growth periods are there?

d) Write an equation that models the growth of the investment, and use it to determine the value of the investment after 15 years.

6. A species of bacteria has a population of 500 at noon. It doubles every 10 h. **K** The function that models the growth of the population, P, at any hour, t, is

$$P(t) = 500\left(2^{\frac{t}{10}}\right).$$

a) Why is the exponent $\frac{t}{10}$?
b) Why is the base 2?
c) Why is the multiplier 500?
d) Determine the population at midnight.
e) Determine the population at noon the next day.
f) Determine the time at which the population first exceeds 2000.

7. Which of these functions describe exponential decay? Explain.

a) $g(x) = -4(3)^x$
b) $h(x) = 0.8(1.2)^x$
c) $j(x) = 3(0.8)^{2x}$
d) $k(x) = \frac{1}{3}(0.9)^{\frac{x}{2}}$

8. A town with a population of 12 000 has been growing at an average rate of 2.5% for the last 10 years. Suppose this growth rate will be maintained in the future. The function that models the town's growth is

$$P(n) = 12(1.025^n)$$

where $P(n)$ represents the population (in thousands) and n is the number of years from now.

a) Determine the population of the town in 10 years.
b) Determine the number of years until the population doubles.
c) Use this equation (or another method) to determine the number of years ago that the population was 8000. Answer to the nearest year.
d) What are the domain and range of the function?

9. A student records the internal temperature of a hot sandwich that has been **A** left to cool on a kitchen counter. The room temperature is 19 °C. An equation that models this situation is

$$T(t) = 63(0.5)^{\frac{t}{10}} + 19$$

where T is the temperature in degrees Celsius and t is the time in minutes.

a) What was the temperature of the sandwich when she began to record its temperature?
b) Determine the temperature, to the nearest degree, of the sandwich after 20 min.
c) How much time did it take for the sandwich to reach an internal temperature of 30 °C?

10. In each case, write an equation that models the situation described. Explain what each part of each equation represents.

a) the percent of colour left if blue jeans lose 1% of their colour every time they are washed

b) the population if a town had 2500 residents in 1990 and grew at a rate of 0.5% each year after that for t years

c) the population of a colony if a single bacterium takes 1 day to divide into two; the population is P after t days

11. A population of yeast cells can double in as little as 1 h. Assume an initial population of 80 cells.

a) What is the growth rate, in percent per hour, of this colony of yeast cells?

b) Write an equation that can be used to determine the population of cells at t hours.

c) Use your equation to determine the population after 6 h.

d) Use your equation to determine the population after 90 min.

e) Approximately how many hours would it take for the population to reach 1 million cells?

f) What are the domain and range for this situation?

12. A collector's hockey card is purchased in 1990 for $5. The value increases by 6% every year.

a) Write an equation that models the value of the card, given the number of years since 1990.

b) Determine the increase in value of the card in the 4th year after it was purchased (from year 3 to year 4).

c) Determine the increase in value of the card in the 20th year after it was purchased.

13. Light intensity in a lake falls by 9% per metre of depth relative to the surface.

a) Write an equation that models the intensity of light per metre of depth. Assume that the intensity is 100% at the surface.

b) Determine the intensity of light at a depth of 7.5 m.

14. A disinfectant is advertised as being able to kill 99% of all germs with each application.

a) Write an equation that represents the percent of germs left with n applications.

b) Suppose a kitchen countertop has 10 billion (10^{10}) germs. How many applications are required to eliminate all of the germs?

15. **T** A town has a population of 8400 in 1990. Fifteen years later, its population grew to 12 500. Determine the average annual growth rate of this town's population.

16. **C** A group of yeast cells grows by 75% every 3 h. At 9 a.m., there are 200 yeast cells.

a) Write an equation that models the number of cells, given the number of hours after 9 a.m.

b) Explain how each part of your equation is related to the given information.

Extending

17. In the year 2002, a single baby girl born in Alberta was given the name Nevaeh.
- Two years later, there were 18 girls (including the first one) with that name.
- By 2005, there were 70 girls with the name (*National Post,* Wed., May 24, 2006, p. A2).

a) Investigate whether or not this is an example of exponential growth.
b) Determine what the growth rate might be, and create a possible equation to model the growth in the popularity of this name.
c) Discuss any limitations of your model.

18. Psychologist H. Ebbinghaus performed experiments in which he had people memorize lists of words and then tested their memory of the list. He found that the percent of words they remembered can be modelled by

$$R(T) = \frac{100}{1 + 1.08T^{0.21}}$$

where $R(T)$ is the percent of words remembered after T hours. This equation is now known as the "forgetting curve," even though it actually models the percent of words remembered!

a) Graph this function with technology. Describe its features and decide whether or not it is an example of exponential decay.
b) Predict the percent of words remembered after 24 h.

Curious | *Math*

Zeno's Paradox

Zeno of Elea (c. 490–425 BCE), a Greek philosopher and mathematician, is famous for his paradoxes that deal with motion. (A paradox is a statement that runs counter to common sense, but may actually be true.) Zeno suggested that it is impossible to get to point *B* from point *A*.

He illustrated his point of view with a story.

Achilles (point *A*) and Tortoise agreed to have a race. Tortoise was given a head start (point *B*). After the race started, Achilles travelled half the distance between himself and Tortoise (point *C*). And again, after a period of time, he travelled half the remaining distance between himself and Tortoise (point *D*). Each time he arrived at the halfway point, there was a new, and smaller, halfway point. So if you look at it this way, there is an infinite number of halfway points and Achilles would never catch up to Tortoise.

1. What is the function that models this problem, if Tortoise was given a head start of 1000 m? Does this function support Zeno's paradox? Explain.

2. Who will win the race between Achilles and Tortoise? Explain.

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