4.6 Transformations of Exponential Functions

YOU WILL NEED

- graphing calculator

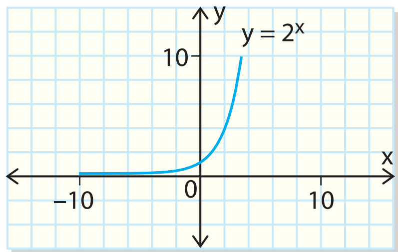
GOAL

Investigate the effects of transformations on the graphs and equations of exponential functions.

INVESTIGATE the Math

Recall the graph of the function $f(x) = 2^x$.

- It is an increasing function.
- It has a y-intercept of 1.
- Its asymptote is the line $y = 0$.



? If $f(x) = 2^x$, how do the parameters a, k, d, and c in the function $g(x) = af(k(x - d)) + c$ affect the size and shape of the graph of $f(x)$?

Tech Support

You can adjust to these settings by pressing ZOOM and 4 . ZDecimal.

A. Use your graphing calculator to graph the function $f(x) = 2^x$. Use the window settings shown.

B. Predict what will happen to the function $f(x) = 2^x$ if it is changed to

- $g(x) = 2^x + 1$ or $h(x) = 2^x - 1$
- $p(x) = 2^{x+1}$ or $q(x) = 2^{x-1}$

C. Copy and complete the table by graphing the given functions, one at a time, as Y2. Keep the graph of $f(x) = 2^x$ as Y1 for comparison. For each function, sketch the graph on the same grid and describe how its points and features have changed.

<table>
<tr><th>Function</th><th>Sketch</th><th>Table of Values</th><th>Description of Changes of New Graph</th></tr>
<tr><td>$g(x) = 2^x + 1$</td><td></td><td><table><tr><th>x</th><th>$y = 2^x$</th><th>$y = 2^x + 1$</th></tr><tr><td>-1</td><td></td><td></td></tr><tr><td>0</td><td></td><td></td></tr><tr><td>1</td><td></td><td></td></tr></table></td><td></td></tr>
<tr><td>$h(x) = 2^x - 1$</td><td></td><td><table><tr><th>x</th><th>$y = 2^x$</th><th>$y = 2^x - 1$</th></tr><tr><td>-1</td><td></td><td></td></tr><tr><td>0</td><td></td><td></td></tr><tr><td>1</td><td></td><td></td></tr></table></td><td></td></tr>
<tr><td>$p(x) = 2^{x+1}$</td><td></td><td><table><tr><th>x</th><th>$y = 2^x$</th><th>$y = 2^{x+1}$</th></tr><tr><td>-1</td><td></td><td></td></tr><tr><td>0</td><td></td><td></td></tr><tr><td>1</td><td></td><td></td></tr></table></td><td></td></tr>
<tr><td>$q(x) = 2^{x-1}$</td><td></td><td><table><tr><th>x</th><th>$y = 2^x$</th><th>$y = 2^{x-1}$</th></tr><tr><td>-1</td><td></td><td></td></tr><tr><td>0</td><td></td><td></td></tr><tr><td>1</td><td></td><td></td></tr></table></td><td></td></tr>
</table>

D. Describe the types of transformations you observed in part C. Comment on how the features and points of the original graph were changed by the transformations.

E. Predict what will happen to the function $f(x) = 2^x$ if it is changed to

- $g(x) = 3(2^x)$
- $h(x) = 0.5(2^x)$
- $j(x) = -(2^x)$

F. Create a table like the one in part C using the given functions in part E. Graph each function one at a time, as Y2. Keep the graph of $f(x) = 2^x$ as Y1 for comparison. In your table, sketch the graph on the same grid, complete the table of values, and describe how its points and features have changed.

G. Describe the types of transformations you observed in part F. Comment on how the features and points of the original graph were changed by the transformations.

H. Predict what will happen to the function $f(x) = 2^x$ if it is changed to
- $g(x) = 2^{2x}$
- $h(x) = 2^{0.5x}$
- $j(x) = 2^{-x}$

I. Create a table like the one in part C using the given functions in part H. Graph each function one at a time, as Y2. Keep the graph of $f(x) = 2^x$ as Y1 for comparison. In your table, sketch the graph, complete the table of values, and describe how its points and features have changed.

J. Describe the types of transformations you observed in part I. Comment on how the features and points of the original graph were changed by such transformations.

K. Choose several different bases for the original function. Experiment with different kinds of transformations. Are the changes in the function affected by the value of the base?

L. Summarize your findings by describing the roles that the parameters a, k, d, and c play in the function defined by $f(x) = ab^{k(x-d)} + c$.

Reflecting

M. Which transformations change the shape of the curve? Explain how the equation is changed by these transformations.

N. Which transformations change the location of the asymptote? Explain how the equation is changed by these transformations.

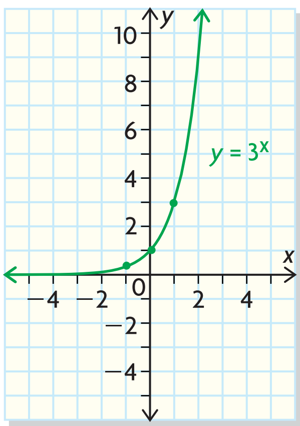
O. Do the transformations affect $f(x) = b^x$ in the same way they affect $f(x) = x$, $f(x) = x^2$, $f(x) = \frac{1}{x}$, $f(x) = \sqrt{x}$, and $f(x) = |x|$? Explain.

APPLY the Math

EXAMPLE 1 Using reasoning to predict the shape of the graph of an exponential function

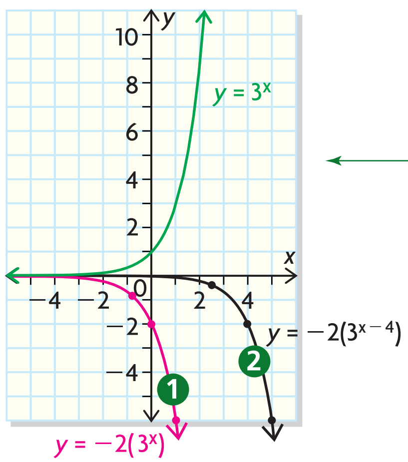
Use transformations to sketch the function $y = -2(3^{x-4})$. State the domain and range.

J.P.'s Solution



I began by sketching the graph of $y = 3^x$.

Three of its key points are (0, 1), (1, 3), and $(-1, \frac{1}{3})$. The asymptote is the x-axis, $y = 0$.



The function I really want to graph is $y = -2(3^{x-4})$. The base function, $y = 3^x$, was changed by multiplying all y-values by -2, resulting in a vertical stretch of factor 2 and a reflection in the x-axis.

Subtracting 4 from x results in a translation of 4 units to the right.

I could perform these two transformations in either order, since one affected only the x-coordinate and the other affected only the y-coordinate. I did the stretch first.

1. With vertical stretches and reflection in the x-axis (multiplying by -2, graphed in red), my key points had their y-values doubled:

   $(0, 1) \rightarrow (0, -2)$, $(1, 3) \rightarrow (1, -6)$, and $(-1, \frac{1}{3}) \rightarrow (-1, -\frac{2}{3})$

   The asymptote $y = 0$ was not affected.

2. With translations (subtracting 4, graphed in black), the key points changed by adding 4 to the x-values:

   $(0, -2) \rightarrow (4, -2)$, $(1, -6) \rightarrow (5, -6)$, and $(-1, -\frac{2}{3}) \rightarrow (3, -\frac{2}{3})$

   This shifted the curve 4 units to the right. The asymptote $y = 0$ was not affected.

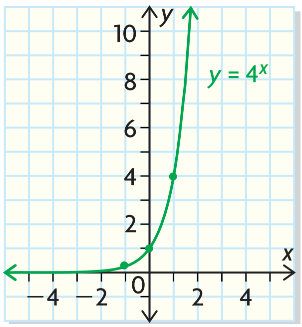
The domain of the original function, $\{x \in \mathbf{R}\}$, was not changed by the transformations.

The range, determined by the equation of the asymptote, was $y > 0$ for the original function. There was no vertical translation, so the asymptote remained the same, but, due to the reflection in the x-axis, the range changed to $\{y \in \mathbf{R} \mid y < 0\}$.

EXAMPLE 2 Connecting the graphs of different exponential functions

Use transformations to sketch the graph of $y = 4^{-2x-4} + 3$.

Ilia's Solution



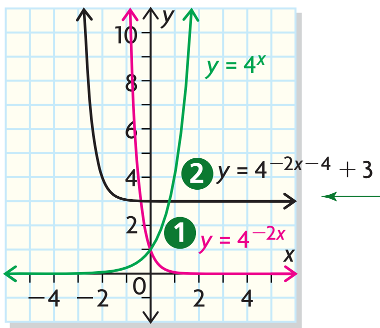
I began by sketching the graph of the base curve, $y = 4^x$. It has the line $y = 0$ as its asymptote, and three of its key points are (0, 1), (1, 4), and $(-1, \frac{1}{4})$.

I factored the exponent to see the different transformations clearly:

$y = 4^{-2(x+2)} + 3$

The x-values were multiplied by -2, resulting in a horizontal compression of factor $\frac{1}{2}$, as well as a reflection in the y-axis.

There were two translations: 2 units to the left and 3 units up.



I applied the transformations in the proper order.

The table shows how the key points and the equation of the asymptote change:

Point or Asymptote	Horizontal Stretch and Reflection	Horizontal Translation	Vertical Translation
(0, 1)	(0, 1)	$(-2, 1)$	$(-2, 4)$
(1, 4)	$(-\frac{1}{2}, 4)$	$(-2\frac{1}{2}, 4)$	$(-2\frac{1}{2}, 7)$
$(-1, \frac{1}{4})$	$(\frac{1}{2}, \frac{1}{4})$	$(-1\frac{1}{2}, \frac{1}{4})$	$(-1\frac{1}{2}, 3\frac{1}{4})$
$y = 0$	$y = 0$	$y = 0$	$y = 3$

1. There was one stretch and one reflection, each of which applied only to the x-coordinate: a horizontal compression of factor $\frac{1}{2}$ and a reflection in the y-axis (shown in red).
2. There were two translations: 2 units to the left and 3 units up (shown in black).

EXAMPLE 3 Communicating the relationship among different exponential functions

Compare and contrast the functions defined by $f(x) = 9^x$ and $g(x) = 3^{2x}$.

Pinder's Solution: Using Exponent Rules

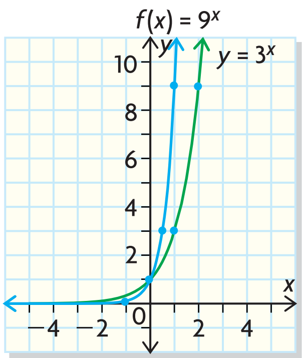
$$\begin{aligned} f(x) &= 9^x \\ &= (3^2)^x \\ &= 3^{2x} \\ &= g(x) \end{aligned}$$

Both functions are the same.

9 is a power of 3, so, to make it easier to compare 9^x with 3^{2x}, I substituted 3^2 for 9 in the first equation.

By the power-of-a-power rule, $f(x)$ has the same equation as $g(x)$.

Kareem's Solution



Both functions are the same.

$f(x) = 9^x$ is an exponential function with a y-intercept of 1 and the line $y = 0$ as its asymptote. Also, $f(x) = 9^x$ passes through the points (1, 9) and $\left(-1, \frac{1}{9}\right)$.

$g(x) = 3^{2x}$ is the base function $y = 3^x$ after a horizontal compression of factor $\frac{1}{2}$. This means that the key points change by multiplying their x-values by $\frac{1}{2}$. The point (1, 3) becomes (0.5, 3) and (2, 9) becomes (1, 9). When I plotted these points, I got points on the curve of $f(x)$.

EXAMPLE 4 Connecting the verbal and algebraic descriptions of transformations of an exponential curve

An exponential function with a base of 2 has been stretched vertically by a factor of 1.5 and reflected in the y-axis. Its asymptote is the line $y = 2$. Its y-intercept is (0, 3.5). Write an equation of the function and discuss its domain and range.

Louise's Solution

$y = a2^{k(x-d)} + c$

I began by writing the general form of the exponential equation with a base of 2.

$y = 1.5(2^{-x}) + c$

Since the function had been stretched vertically by a factor of 1.5, $a = 1.5$. The function has also been reflected in the y-axis, so $k = -1$. There was no horizontal translation, so $d = 0$.

$y = 1.5(2^{-x}) + 2$

Since the horizontal asymptote is $y = 2$ the function has been translated vertically by 2 units, so $c = 2$.

$$
\begin{aligned}
y &= 1.5(2^{-(0)}) + 2 \\
&= 1.5(1) + 2 \\
&= 3.5
\end{aligned}
$$

I substituted $x = 0$ into the equation and calculated the y-intercept. It matched the stated y-intercept, so my equation seemed to represent this function.

The original domain is $\{x \in \mathbf{R}\}$. The transformations didn't change this.

The range changed, since there was a vertical translation. The asymptote moved up 2 units along with the function, so the range is $\{y \in \mathbf{R} \mid y > 2\}$.

In Summary

Key Ideas

- In functions of the form $g(x) = af(k(x - d)) + c$, the constants a, k, d, and c change the location or shape of the graph of $f(x)$. The shape is dependent on the value of the base function $f(x) = b^x$, as well as on the values of a and k.
- Functions of the form $g(x) = af(k(x - d)) + c$ can be graphed by applying the appropriate transformations to the key points of the base function $f(x) = b^x$, one at a time, following the order of operations. The horizontal asymptote changes when vertical translations are applied.

Need to Know

- In exponential functions of the form $g(x) = a\,b^{k(x-d)} + c$:
  - If $|a| > 1$, a vertical stretch by a factor of $|a|$ occurs. If $0 < |a| < 1$, a vertical compression by a factor of $|a|$ occurs. If a is also negative, then the function is reflected in the x-axis.
  - If $|k| > 1$, a horizontal compression by a factor of $|\frac{1}{k}|$ occurs. If $0 < |k| < 1$, a horizontal stretch by a factor of $|\frac{1}{k}|$ occurs. If k is also negative, then the function is reflected in the y-axis.
  - If $d > 0$, a horizontal translation of d units to the right occurs. If $d < 0$, a horizontal translation to the left occurs.
  - If $c > 0$, a vertical translation of c units up occurs. If $c < 0$, a vertical translation of c units down occurs.
  - You might have to factor the exponent to see what the transformations are. For example, if the exponent is $2x + 2$, it is easier to see that there was a horizontal stretch of 2 and a horizontal translation of 1 to the left if you factor to $2(x + 1)$.
  - When transforming functions, consider the order. You might perform stretches and reflections followed by translations, but if the stretch involves a different coordinate than the translation, the order doesn't matter.
  - The domain is always $\{x \in \mathbf{R}\}$. Transformations do not change the domain.
  - The range depends on the location of the horizontal asymptote and whether the function is above or below the asymptote. If it is above the asymptote, its range is $y > c$. If it is below, its range is $y < c$.

CHECK Your Understanding

1. Each of the following are transformations of $f(x) = 3^x$. Describe each transformation.

a) $g(x) = 3^x + 3$

b) $g(x) = 3^{x+3}$

c) $g(x) = \frac{1}{3}(3^x)$

d) $g(x) = 3^{\frac{x}{3}}$

2. For each transformation, state the base function and then describe the transformations in the order they could be applied.

a) $f(x) = -3(4^{x+1})$

b) $g(x) = 2\left(\frac{1}{2}\right)^{2x} + 3$

c) $h(x) = 7(0.5^{x-4}) - 1$

d) $k(x) = 5^{3x-6}$

3. State the y-intercept, the equation of the asymptote, and the domain and range for each of the functions in questions 1 and 2.

PRACTISING

4. Each of the following are transformations of $h(x) = \left(\frac{1}{2}\right)^x$. Use words to describe the sequence of transformations in each case.

a) $g(x) = -\left(\frac{1}{2}\right)^{2x}$

b) $g(x) = 5\left(\frac{1}{2}\right)^{-(x-3)}$

c) $g(x) = -4\left(\frac{1}{2}\right)^{3x+9} - 6$

5. Let $f(x) = 4^x$. For each function that follows,

K
- state the transformations that must be applied to $f(x)$
- state the y-intercept and the equation of the asymptote
- sketch the new function
- state the domain and range

a) $g(x) = 0.5f(-x) + 2$
b) $h(x) = -f(0.25x + 1) - 1$
c) $g(x) = -2f(2x - 6)$
d) $h(x) = f(-0.5x + 1)$

6. Compare the functions $f(x) = 6^x$ and $g(x) = 3^{2x}$.

7. A cup of hot liquid was left to cool in a room whose temperature was 20 °C.
C The temperature changes with time according to the function $T(t) = 80\left(\frac{1}{2}\right)^{\frac{t}{30}} + 20$. Use your knowledge of transformations to sketch this function. Explain the meaning of the y-intercept and the asymptote in the context of this problem.

8. The doubling time for a certain type of yeast cell is 3 h. The number of cells after t hours is described by $N(t) = N_0 2^{\frac{t}{3}}$, where N_0 is the initial population.

a) How would the graph and the equation change if the doubling time were 9 h?

b) What are the domain and range of this function in the context of this problem?

9. Match the equation of the functions from the list to the appropriate graph at the top of the next page.

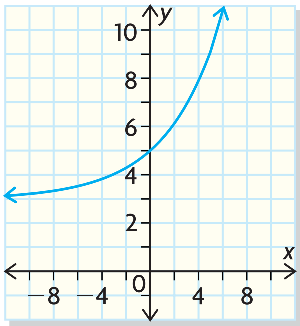
a) $f(x) = -\left(\frac{1}{4}\right)^{-x} + 3$

b) $y = \left(\frac{1}{4}\right)^{x} + 3$

c) $g(x) = -\left(\frac{5}{4}\right)^{-x} + 3$

d) $h(x) = 2\left(\frac{5}{4}\right)^{x} + 3$

i)



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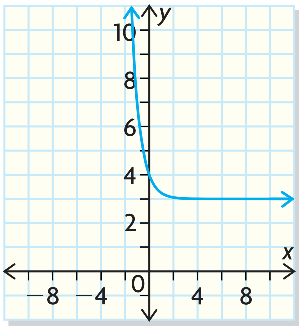

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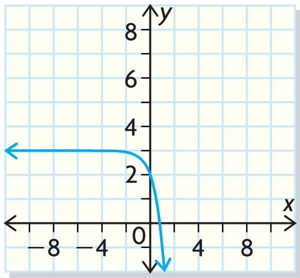

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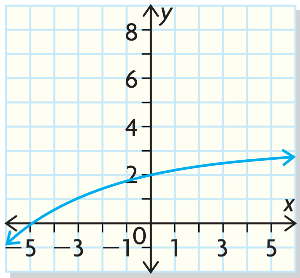
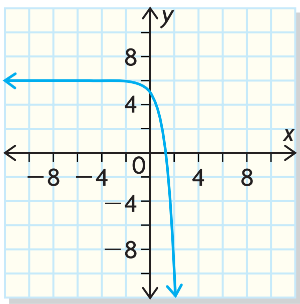
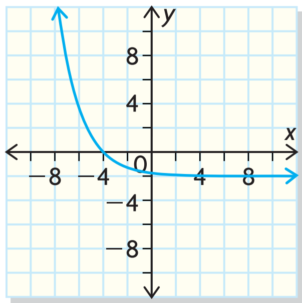


10. Each graph represents a transformation of the function $f(x) = 2^x$. Write an equation for each one.

a)



b)



11. State the transformations necessary (and in the proper order) to transform
T $f(x) = 2^{x+1} + 5$ to $g(x) = \frac{1}{4}(2^x)$.

Extending

12. Use your knowledge of transformations to sketch the function
$f(x) = \frac{-3}{2^{x+2}} - 1$.

13. Use your knowledge of transformations to sketch the function
$g(x) = 4 - 2\left(\frac{1}{3}\right)^{-0.5x+1}$.

14. State the transformations necessary (and in the proper order) to transform
$m(x) = -\left(\frac{3}{2}\right)^{2x-2}$ to $n(x) = -\left(\frac{9}{4}\right)^{-x+1} + 2$.