4.4 Simplifying Algebraic Expressions Involving Exponents

GOAL

Simplify algebraic expressions involving powers and radicals.

LEARN ABOUT the Math

The ratio of the surface area to the volume of microorganisms affects their ability to survive. An organism with a higher surface area-to-volume ratio is more buoyant and uses less of its own energy to remain near the surface of a liquid, where food is more plentiful.

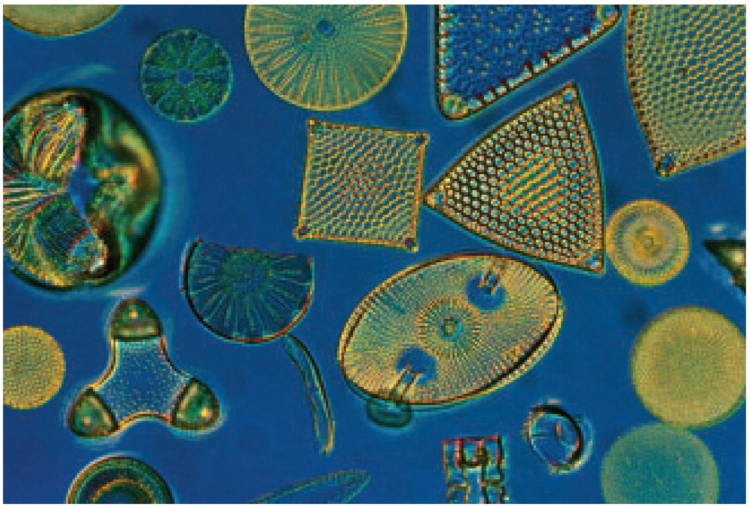
Mike is calculating the surface area-to-volume ratio for different-sized cells. He assumes that the cells are spherical.

For a sphere,

$$SA(r) = 4\pi r^2 \quad \text{and} \quad V(r) = \tfrac{4}{3}\pi r^3.$$

He substitutes the value of the radius into each formula and then divides the two expressions to calculate the ratio.

Radius (μm)	Surface Area/ Volume
1	$\frac{4\pi}{\left(\frac{4}{3}\pi\right)}$
1.5	$\frac{9\pi}{4.5\pi}$
2	$\frac{16\pi}{\left(\frac{32}{3}\pi\right)}$
2.5	$\frac{25\pi}{\left(\frac{125}{6}\pi\right)}$
3	$\frac{36\pi}{36\pi}$
3.5	$\frac{49\pi}{\left(\frac{343}{6\pi}\right)}$



? How can Mike simplify the calculation he uses?

EXAMPLE 1 Representing the surface area-to-volume ratio

Simplify $\frac{SA(r)}{V(r)}$, given that $SA(r) = 4\pi r^2$ and $V(r) = \frac{4}{3}\pi r^3$.

Bram's Solution

$\frac{SA(r)}{V(r)}$ ← I used the formulas for *SA* and *V* and wrote the ratio.

$= \frac{4\pi r^2}{\frac{4}{3}\pi r^3}$ ← The numerator and denominator have a factor of π, so I divided both by π.

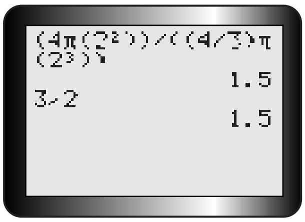
I started to simplify the expression by dividing the coefficients.

$\left(4 \div \frac{4}{3} = 4 \times \frac{3}{4} = 3\right)$

$= 3r^{-1}$

$= \frac{3}{r}$

The bases of the powers were the same, so I subtracted exponents to simplify the part of the expression involving *r*.



← I used a calculator and substituted $r = 2$ in the unsimplified ratio first and my simplified expression next.

Each version gave me the same answer, so I think that they are equivalent, but the second one took far fewer keystrokes!

Reflecting

A. How can you use the simplified ratio to explain why the values in Mike's table kept decreasing?

B. Is it necessary to simplify an algebraic expression before you substitute numbers and perform calculations? Explain.

C. What are the advantages and disadvantages to simplifying an algebraic expression prior to performing calculations?

D. Do the exponent rules used on algebraic expressions work the same way as they do on numerical expressions? Explain by referring to Bram's work.

APPLY the Math

EXAMPLE 2 Connecting the exponent rules to the simplification of algebraic expressions

Simplify $\frac{(2x^{-3}y^2)^3}{(x^3y^{-4})^2}$.

Adnan's Solution

$$\frac{(2x^{-3}y^2)^3}{(x^3y^{-4})^2} = \frac{(2)^3(x^{-3})^3(y^2)^3}{(x^3)^2(y^{-4})^2}$$

I used the product-of-powers rule to raise each factor in the numerator to the third power and to square each factor in the denominator. Then I multiplied exponents.

$$= \frac{8x^{-9}y^6}{x^6y^{-8}}$$
$$= 8x^{-9-6}y^{6-(-8)}$$
$$= 8x^{-15}y^{14}$$
$$= \frac{8y^{14}}{x^{15}}$$

I simplified the whole expression by subtracting exponents of terms with the same base.

One of the powers had a negative exponent. To write it with positive exponents, I used its reciprocal.

EXAMPLE 3 Selecting a computational strategy to evaluate an expression

Evaluate the expression $\frac{(x^{2n+1})(x^{3n-1})}{x^{2n-5}}$ for $x = -3$ and $n = 2$.

Bonnie's Solution: Substituting, then Simplifying

$$\frac{(x^{2n+1})(x^{3n-1})}{x^{2n-5}} = \frac{(-3)^{2(2)+1}(-3)^{3(2)-1}}{(-3)^{2(2)-5}}$$
$$= \frac{(-3)^5(-3)^5}{(-3)^{-1}}$$
$$= \frac{(-243)(-243)}{\frac{1}{-3}}$$
$$= -177\ 147$$

I substituted the values for x and n into the expression.

Then I evaluated the numerator and denominator separately, before dividing one by the other.

Alana's Solution: Simplifying, then Substituting

$$\frac{(x^{2n+1})(x^{3n-1})}{x^{2n-5}}$$

Each power had the same base, so I simplified by using exponent rules before I substituted.

$$= \frac{x^{(2n+1)+(3n-1)}}{x^{2n-5}}$$

I added the exponents in the numerator to express it as a single power.

$$= \frac{x^{5n}}{x^{2n-5}}$$

$$= x^{(5n)-(2n-5)}$$

Then I subtracted the exponents in the denominator to divide the powers.

$$= x^{3n+5}$$

$$= (-3)^{3(2)+5}$$

$$= (-3)^{11}$$

$$= -177\ 147$$

Once I had a single power, I substituted -3 for x and 2 for n and evaluated.

EXAMPLE 4 Simplifying an expression involving powers with rational exponents

Simplify $\dfrac{(27a^{-3}b^{12})^{\frac{1}{3}}}{(16a^{-8}b^{12})^{\frac{1}{2}}}$.

Jane's Solution

$$\frac{(27a^{-3}b^{12})^{\frac{1}{3}}}{(16a^{-8}b^{12})^{\frac{1}{2}}} = \frac{27^{\frac{1}{3}}a^{\frac{-3}{3}}b^{\frac{12}{3}}}{16^{\frac{1}{2}}a^{\frac{-8}{2}}b^{\frac{12}{2}}}$$

$$= \frac{3a^{-1}b^{4}}{4a^{-4}b^{6}}$$

In the numerator, I applied the exponent $\frac{1}{3}$ to each number or variable inside the parentheses, using the power-of-a-power rule. I did the same in the denominator, applying the exponent $\frac{1}{2}$ to the numbers and variables.

$$= \frac{3}{4}a^{-1+4}b^{4-6}$$

I simplified by subtracting the exponents.

$$= \frac{3}{4}a^{3}b^{-2}$$

$$= \frac{3a^{3}}{4b^{2}}$$

I expressed the answer with positive exponents.

Sometimes it is necessary to express an expression involving radicals using exponents in order to simplify it.

EXAMPLE 5 Representing an expression involving radicals as a single power

Simplify $\left(\frac{\sqrt[5]{x^8}}{\sqrt{x^3}}\right)^3$.

Albino's Solution

$$\left(\frac{\sqrt[5]{x^8}}{\sqrt{x^3}}\right)^3 = \left(\frac{x^{\frac{8}{5}}}{x^{\frac{3}{2}}}\right)^3$$

Since this is a fifth root divided by a square root, I couldn't write it as a single radical.

$$= \left(x^{\frac{8}{5}-\frac{3}{2}}\right)^3$$

$$= \left(x^{\frac{16}{10}-\frac{15}{10}}\right)^3$$

$$= \left(x^{\frac{1}{10}}\right)^3$$

$$= x^{\frac{3}{10}}$$

$$= \sqrt[10]{x^3}$$

I changed the radical expressions to exponential form and used exponent rules to simplify.

When I got a single power, I converted it to radical form.

In Summary

Key Idea

- Algebraic expressions involving powers containing integer and rational exponents can be simplified with the use of the exponent rules in the same way numerical expressions can be simplified.

Need to Know

- When evaluating an algebraic expression by substitution, simplify prior to substituting. The answer will be the same if substitution is done prior to simplifying, but the number of calculations will be reduced.
- Algebraic expressions involving radicals can often be simplified by changing the expression into exponential form and applying the rules for exponents.

CHECK Your Understanding

1. Simplify. Express each answer with positive exponents.

a) $x^4(x^3)$

b) $(p^{-3})(p)^5$

c) $\frac{m^5}{m^{-3}}$

d) $\frac{a^{-4}}{a^{-2}}$

e) $(y^3)^2$

f) $(k^6)^{-2}$

2. Simplify. Express each answer with positive exponents.

a) $y^{10}(y^4)^{-3}$

b) $(x^{-3})^{-3}(x^{-1})^5$

c) $\dfrac{(n^{-4})^3}{(n^{-3})^{-4}}$

d) $\dfrac{w^4(w^{-3})}{(w^{-2})^{-1}}$

e) $\dfrac{(x^{-1})^4x}{x^{-3}}$

f) $\dfrac{(b^{-7})^2}{b(b^{-5})b^9}$

3. Consider the expression $\dfrac{x^7(y^2)^3}{x^5y^4}$.

a) Substitute $x = -2$ and $y = 3$ into the expression, and evaluate it.

b) Simplify the expression. Then substitute the values for x and y to evaluate it.

c) Which method seems more efficient?

PRACTISING

4. Simplify. Express answers with positive exponents.

a) $(pq^2)^{-1}(p^3q^3)$

b) $\left(\dfrac{x^3}{y}\right)^{-2}$

c) $\dfrac{(ab)^{-2}}{b^5}$

d) $\dfrac{m^2n^2}{(m^3n^{-2})^2}$

e) $\dfrac{(w^2x)^2}{(x^{-1})^2w^3}$

f) $\left(\dfrac{(ab)^{-1}}{a^2b^{-3}}\right)^{-2}$

5. Simplify. Express answers with positive exponents.

a) $(3xy^4)^2(2x^2y)^3$

b) $\dfrac{(2a^3)^2}{4ab^2}$

c) $\dfrac{(10x)^{-1}y^3}{15x^3y^{-3}}$

d) $\dfrac{(3m^4n^2)^2}{12m^{-2}n^6}$

e) $\dfrac{p^{-5}(r^3)^2}{(p^2r)^2(p^{-1})^2}$

f) $\left(\dfrac{(x^3y)^{-1}(x^4y^3)}{(x^2y^{-3})^{-2}}\right)^{-1}$

6. Simplify. Express answers with positive exponents.

K

a) $(x^4)^{\frac{1}{2}}(x^6)^{-\frac{1}{3}}$

b) $\dfrac{9(c^8)^{0.5}}{(16c^{12})^{0.25}}$

c) $\dfrac{\sqrt{25m^{-12}}}{\sqrt{36m^{10}}}$

d) $\sqrt[3]{\dfrac{(10x^3)^2}{(10x^6)^{-1}}}$

e) $\left(\dfrac{(32x^5)^{-2}}{(x^{-1})^{10}}\right)^{0.2}$

f) $\dfrac{\sqrt[10]{1024x^{20}}}{\sqrt[9]{512x^{27}}}$

7. Evaluate each expression. Express answers in rational form with positive exponents.

a) $(16x^6y^4)^{\frac{1}{2}}$ for $x = 2, y = 1$

b) $\dfrac{(9p^{-2})^{\frac{1}{2}}}{6p^2}$ for $p = 3$

c) $\dfrac{(81x^4y^6)^{\frac{1}{2}}}{8(x^9y^3)^{\frac{1}{3}}}$ for $x = 10, y = 5$

d) $\left(\dfrac{(25a^4)^{-1}}{(7a^{-2}b)^2}\right)^{\frac{1}{2}}$ for $a = 11, b = 10$

8. Evaluate. Express answers in rational form with positive exponents.

a) $(\sqrt{10\,000x})^{\frac{3}{2}}$ for $x = 16$

b) $\left(\frac{(4x^3)^4}{(x^3)^6}\right)^{-0.5}$ for $x = 5$

c) $(-2a^2b)^{-3}\sqrt{25a^4b^6}$ for $a = 1, b = 2$

d) $\sqrt{\frac{(18m^{-5}n^2)(32m^2n)}{4mn^{-3}}}$ for $m = 10, n = 1$

9. Simplify. Express answers in rational form with positive exponents.

a) $(36m^4n^6)^{0.5}(81m^{12}n^8)^{0.25}$

b) $\left(\frac{(6x^3)^2(6y^3)}{(9xy)^6}\right)^{-\frac{1}{3}}$

c) $\left(\frac{\sqrt{64a^{12}}}{(a^{1.5})^{-6}}\right)^{\frac{2}{3}}$

d) $\left(\frac{(x^{18})^{\frac{-1}{6}}}{\sqrt[5]{243x^{10}}}\right)^{0.5}$

10. **T** If $M = \frac{(16x^8y^{-4})^{\frac{1}{4}}}{32x^{-2}y^8}$, determine values for x and y so that

a) $M = 1$ **b)** $M > 1$ **c)** $0 < M < 1$ **d)** $M < 0$

11. **A** The volume and surface area of a cylinder are given, respectively, by the formulas

$$V = \pi r^2 h \quad \text{and} \quad SA = 2\pi rh + 2\pi r^2.$$

a) Determine an expression, in simplified form, that represents the surface area-to-volume ratio for a cylinder.

b) Calculate the ratio for a radius of 0.8 cm and a height of 12 cm.

12. If $x = -2$ and $y = 3$, write the three expressions in order from least to greatest.

$$\frac{y^{-4}(x^2)^{-3}y^{-3}}{x^{-5}(y^{-4})^2}, \frac{x^{-3}(y^{-1})^{-2}}{(x^{-5})(y^4)}, (y^{-5})(x^5)^{-2}(y^2)(x^{-3})^{-4}$$

13. **C** How is simplifying algebraic expressions like simplifying numerical ones? How is it different?

Extending

14. a) The formula for the volume of a sphere of radius r is $V(r) = \frac{4}{3}\pi r^3$. Solve this equation for r. Write two versions, one in radical form and one in exponential form.

b) Determine the radius of a sphere with a volume of $\frac{256\pi}{3}$ m^3.

15. Simplify $\frac{\sqrt{x(x^{2n+1})}}{\sqrt[3]{x^{3n}}}$, $x > 0$.