4.2 Working with Integer Exponents

GOAL

Investigate powers that have integer or zero exponents.

LEARN ABOUT the Math

The metric system of measurement is used in most of the world. A key feature of the system is its ease of use. Since all units differ by multiples of 10, it is easy to convert from one unit to another. Consider the chart listing the prefix names and their factors for the unit of measure for length, the metre.

Name	Symbol	Multiple of the Metre	Multiple as a Power of 10
terametre	Tm	1 000 000 000 000	10^{12}
gigametre	Gm	1 000 000 000	10^{9}
megametre	Mm	1 000 000	10^{6}
kilometre	km	1 000	10^{3}
hectometre	hm	100	10^{2}
decametre	dam	10	10^{1}
metre	m	1	
decimetre	dm	0.1	
centimetre	cm	0.01	
millimetre	mm	0.001	
micrometre	μm	0.000 1	
nanometre	nm	0.000 01	
picometre	pm	0.000 001	
femtometre	fm	0.000 000 001	
attometre	am	0.000 000 000 001	

? How can powers be used to represent metric units for lengths less than 1 metre?

EXAMPLE 1 Using reasoning to define zero and negative integer exponents

Use the table to determine how multiples of the unit metre that are less than or equal to 1 can be expressed as powers of 10.

Jemila's Solution

Multiples	Powers
1000	10^3
$1000 \div 10 = 100$	$10^3 \div 10 = 10^2$
$100 \div 10 = 10$	$10^2 \div 10 = 10^1$
$10 \div 10 = 1$	$10^1 \div 10 = 10^0$
$1 \div 10 = 0.1$ $= \frac{1}{10}$	$10^0 \div 10 = 10^{-1}$
$0.1 \div 10 = 0.01$ $= \frac{1}{100}$ $= \frac{1}{10^2}$	$10^{-1} \div 10 = 10^{-2}$
$0.01 \div 10 = 0.001$ $= \frac{1}{1000}$ $= \frac{1}{10^3}$	$10^{-2} \div 10 = 10^{-3}$

I think that $x^{-n} = \frac{1}{x^n}$ is the rule for negative exponents.

As I moved down the table, the powers of 10 decreased by 1, while the multiples were divided by 10. To come up with the next row in the table, I divided the multiples and the powers by 10.

If I continue this pattern, I'll get $10^0 = 1$, $10^{-1} = 0.1$, $10^{-2} = 0.01$, etc.

I rewrote each decimal as a fraction and each denominator as a power of 10.

I noticed that $10^0 = 1$ and $10^{-n} = \frac{1}{10^n}$.

I don't think it mattered that the base was 10. The relationship would be true for any base.

EXAMPLE 2 Connecting the concept of an exponent of 0 to the exponent quotient rule

Use the quotient rule to show that $10^0 = 1$.

David's Solution

$$\frac{10^6}{10^6} = 1$$

I can divide any number except 0 by itself to get 1. I used a power of 10.

$$\frac{10^6}{10^6} = 10^{6-6} = 10^0$$

When you divide powers with the same base, you subtract the exponents.

Therefore, $10^0 = 1$.

I applied the rule to show that a power with zero as the exponent must be equal to 1.

Reflecting

A. What type of number results when x^{-n} is evaluated if x is a positive integer and $n > 1$?

B. How is 10^2 related to 10^{-2}? Why do you think this relationship holds for other opposite exponents?

C. Do you think the rules for multiplying and dividing powers change if the powers have negative exponents? Explain.

APPLY the Math

EXAMPLE 3 Representing powers with integer bases in rational form

Evaluate.

a) 5^{-3} **b)** $(-4)^{-2}$ **c)** -3^{-4}

Stergios's Solution

a) $5^{-3} = \frac{1}{5^3}$

$= \frac{1}{125}$

5^{-3} is what you get if you divide 1 by 5^3. I evaluated the power.

b) $(-4)^{-2} = \frac{1}{(-4)^2}$

$= \frac{1}{16}$

$(-4)^{-2}$ is what you get if you divide 1 by $(-4)^2$. Since the negative sign is in the parentheses, the square of the number is positive.

c) $-3^{-4} = -\frac{1}{3^4}$

$= -\frac{1}{81}$

In this case, the negative sign is not inside the parentheses, so the entire power is negative. I knew that $3^{-4} = \frac{1}{3^4}$.

If the base of a power involving a negative exponent is a fraction, it can be evaluated in a similar manner.

Communication *Tip*

Rational numbers can be written in a variety of forms. The term *rational form* means "Write the number as an integer, or as a fraction."

EXAMPLE 4 Representing powers with rational bases as rational numbers

Evaluate $\left(\frac{2}{3}\right)^{-3}$.

Sadira's Solution

$$\left(\frac{2}{3}\right)^{-3} = \frac{1}{\left(\frac{2}{3}\right)^{3}}$$

$\left(\frac{2}{3}\right)^{-3}$ is what you get if you divide 1 by $\left(\frac{2}{3}\right)^{3}$.

$$= \frac{1}{\left(\frac{8}{27}\right)}$$

$$= 1 \times \frac{27}{8}$$

Dividing by a fraction is the same as multiplying by its reciprocal, so I used this to evaluate the power.

$$= \frac{27}{8}$$

EXAMPLE 5 Selecting a strategy for expressions involving negative exponents

Evaluate $\frac{3^5 \times 3^{-2}}{(3^{-3})^2}$.

Kayleigh's Solution: Using Exponent Rules

$$\frac{3^5 \times 3^{-2}}{(3^{-3})^2} = \frac{3^{5+(-2)}}{3^{-3\times 2}}$$

I simplified the numerator and denominator separately. Then I divided the numerator by the denominator. I added exponents for the numerator, multiplied exponents for the denominator, and subtracted exponents for the final calculation.

$$= \frac{3^3}{3^{-6}}$$

$$= 3^{3-(-6)}$$

$$= 3^9$$

$$= 19\ 683$$

Derek's Solution: Using a Calculator

I entered the expression into my calculator. I made sure I used parentheses around the entire numerator and denominator so that the calculator would compute those values before dividing.

Tech Support

For help with evaluating powers on a graphing calculator, see Technical Appendix, B-15.

In Summary

Key Ideas

- An integer base raised to a negative exponent is equivalent to the reciprocal of the same base raised to the opposite exponent.

$$b^{-n} = \frac{1}{b^n}, \quad \text{where} \quad b \neq 0$$

- A fractional base raised to a negative exponent is equivalent to the reciprocal of the same base raised to the opposite exponent.

$$\left(\frac{a}{b}\right)^{-n} = \frac{1}{\left(\frac{a}{b}\right)^n} = \left(\frac{b}{a}\right)^n, \quad \text{where} \quad a \neq 0, b \neq 0$$

- A number (or expression), other than 0, raised to the power of zero is equal to 1.

$b^0 = 1$, where $b \neq 0$

Need to Know

- When multiplying powers with the same base, add exponents.

$$b^m \times b^n = b^{m+n}$$

- When dividing powers with the same base, subtract exponents.

$$b^m \div b^n = b^{m-n} \quad \text{if} \quad b \neq 0$$

- To raise a power to a power, multiply exponents.

$$(b^m)^n = b^{mn}$$

- In simplifying numerical expressions involving powers, it is customary to present the answer as an integer, a fraction, or a decimal.
- In simplifying algebraic expressions involving powers, it is customary to present the answer with positive exponents.

CHECK Your Understanding

1. Rewrite each expression as an equivalent expression with a positive exponent.

a) 5^{-4} **c)** $\frac{1}{2^{-4}}$ **e)** $\left(\frac{3}{11}\right)^{-1}$

b) $\left(-\frac{1}{10}\right)^{-3}$ **d)** $-\left(\frac{6}{5}\right)^{-3}$ **f)** $\frac{7^{-2}}{8^{-1}}$

2. Write each expression as a single power with a positive exponent.

a) $(-10)^8(-10)^{-8}$ **c)** $\frac{2^8}{2^{-5}}$ **e)** $(-9^4)^{-1}$

b) $6^{-7} \times 6^5$ **d)** $\frac{11^{-3}}{11^5}$ **f)** $[(7^{-3})^{-2}]^{-2}$

3. Which is the greater power, 2^{-5} or $\left(\frac{1}{2}\right)^{-5}$? Explain.

PRACTISING

4. Simplify, then evaluate each expression. Express answers in rational form.

a) $2^{-3}(2^7)$
b) $(-8)^3(-8)^{-3}$
c) $\frac{5^4}{5^6}$
d) $\frac{3^{-8}}{3^{-6}}$
e) $(4^{-3})^{-1}$
f) $(7^{-1})^2$

5. Simplify, then evaluate each expression. Express answers in rational form.

a) $3^3(3^2)^{-1}$
b) $(9 \times 9^{-1})^{-2}$
c) $\frac{(12^{-1})^3}{12^{-3}}$
d) $\frac{(5^3)^{-2}}{5^{-6}}$
e) $(3^{-2}(3^3))^{-2}$
f) $9^7(9^3)^{-2}$

6. Simplify, then evaluate each expression. Express answers in rational form.

a) $10(10^4(10^{-2}))$
b) $8(8^2)(8^{-4})$
c) $\frac{6^{-5}}{(6^2)^{-2}}$
d) $\frac{4^{-10}}{(4^{-4})^3}$
e) $2^8 \times \left(\frac{2^{-5}}{2^6}\right)$
f) $13^{-5} \times \left(\frac{13^2}{13^8}\right)^{-1}$

7. Evaluate. Express answers in rational form.

a) $16^{-1} - 2^{-2}$
b) $(-3)^{-1} + 4^0 - 6^{-1}$
c) $\left(-\frac{2}{3}\right)^{-1} + \left(\frac{2}{5}\right)^{-1}$
d) $\left(\frac{1}{5}\right)^{-1} + \left(-\frac{1}{2}\right)^{-2}$
e) $5^{-3} + 10^{-3} - 8(1000^{-1})$
f) $3^{-2} - 6^{-2} + \frac{3}{2}(-9)^{-1}$

8. Evaluate. Express answers in rational form.

a) $5^2(-10)^{-4}$
b) $16^{-1}(2^5)$
c) $\frac{12^{-1}}{(-4)^{-1}}$
d) $\frac{(-9)^{-2}}{(3^{-1})^2}$
e) $(8^{-1})\left(\frac{2^{-3}}{4^{-1}}\right)$
f) $\frac{(-5)^3(-25)^{-1}}{(-5)^{-2}}$

9. Evaluate. Express answers in rational form.

K
a) $(-4)^{-3}$
b) $(-4)^{-2}$
c) $-(5)^{-3}$
d) $-(5)^{-2}$
e) $(-6)^{-3}$
f) $-(6)^{-2}$

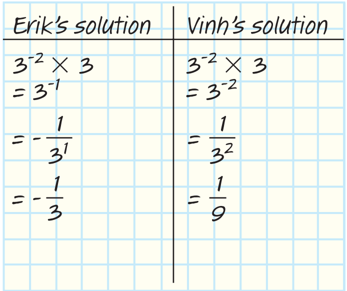
10. Without using your calculator, write the given numbers in order from least to greatest. Explain your thinking.
T

$$(0.1)^{-1}, 4^{-1}, 5^{-2}, 10^{-1}, 3^{-2}, 2^{-3}$$

11. Evaluate each expression for $x = -2$, $y = 3$, and $n = -1$. Express answers in rational form.
A

a) $(x^n + y^n)^{-2n}$
b) $(x^2)^n(y^{-2n})x^{-n}$
c) $\left(\frac{x^n}{y^n}\right)^n$
d) $\left(\frac{xy^n}{(xy)^{2n}}\right)^{2n}$

12. Kendra, Erik, and Vinh are studying. They wish to evaluate $3^{-2} \times 3$. Kendra notices errors in each of her friends' solutions, shown here.



a) Explain where each student went wrong.
b) Create a solution that demonstrates the correct steps.

13. Evaluate using the laws of exponents.

a) $2^3 \times 4^{-2} \div 2^2$ **d)** $4^{-1}(4^2 + 4^0)$ **g)** $\dfrac{3^{-2} \times 2^{-3}}{3^{-1} \times 2^{-2}}$

b) $(2 \times 3)^{-1}$ **e)** $\dfrac{2^5}{3^{-2}} \times \dfrac{3^{-1}}{2^4}$ **h)** $\dfrac{4^{-2} + 3^{-1}}{3^{-2} + 2^{-3}}$

c) $\left(\dfrac{3^{-1}}{2^{-1}}\right)^{-2}$ **f)** $(5^0 + 5^2)^{-1}$ **i)** $\dfrac{5^{-1} - 2^{-2}}{5^{-1} + 2^{-2}}$

14. Find the value of each expression for $a = 1$, $b = 3$, and $c = 2$.

a) ac^c **c)** $(ab)^{-c}$ **e)** $(-a \div b)^{-c}$ **g)** $(a^b b^a)^c$
b) $a^c b^c$ **d)** $(b \div c)^{-a}$ **f)** $(a^{-1}b^{-2})^c$ **h)** $[(b)^{-a}]^{-c}$

15. a) Explain the difference between evaluating $(-10)^3$ and evaluating 10^{-3}.
C **b)** Explain the difference between evaluating $(-10)^4$ and evaluating -10^4.

Extending

16. Determine the exponent that makes each equation true.

a) $16^x = \dfrac{1}{16}$ **c)** $2^x = 1$ **e)** $25^n = \dfrac{1}{625}$

b) $10^x = 0.01$ **d)** $2^n = 0.25$ **f)** $12^n = \dfrac{1}{144}$

17. If $10^{2y} = 25$, determine the value of 10^{-y}, where $y > 0$.

18. Simplify.

a) $(x^2)^{5-r}$ **d)** $x^{3(7-r)}x^r$

b) $(b^{2m+3m}) \div (b^{m-n})$ **e)** $(a^{10-p})\left(\dfrac{1}{a}\right)^P$

c) $(b^{2m+3n}) \div (b^{m-n})$ **f)** $[(3x^4)^{6-m}]\left(\dfrac{1}{x}\right)^m$