3.4 Operations with Radicals

GOAL

Simplify and perform operations on mixed and entire radicals.

INVESTIGATE the Math

The distance, $s(t)$, in millimetres of a particle from a certain point at any time, t, is given by $s(t) = 10\sqrt{4 + t}$. Don needs to find the exact distance between the point and the particle after 20 s. His answer must be in simplest form and he is not permitted to use a decimal.

? What is the exact value of $s(20)$, the distance between the particle and the given point at 20 s?

A. Copy and complete these products of **radicals**:

$\sqrt{25} \times \sqrt{4} = 5 \times 2 = 10$ $\quad\quad$ $\sqrt{25 \times 4} = \sqrt{100} = 10$

$\sqrt{16} \times \sqrt{9} =$ $\quad\quad$ $\sqrt{16 \times 9} = \sqrt{} =$

$\sqrt{4} \times \sqrt{36} =$ $\quad\quad$ $\sqrt{4 \times 36} = \sqrt{} =$

$\sqrt{100} \times \sqrt{9} =$ $\quad\quad$ $\sqrt{100 \times 9} = \sqrt{} =$

radical

a square, cube, or higher root, such as $\sqrt{4} = 2$ or $\sqrt[3]{27} = 3$; $\sqrt{}$ is called the radical symbol

B. Compare the results in each pair of products.

C. Consider $\sqrt{4} \times \sqrt{6}$. From the preceding results, express this product as

a) an **entire radical** $\quad$ **b)** a **mixed radical**

Use your calculator to verify that your products are equivalent.

entire radical

a radical with coefficient 1; for example, $\sqrt{12}$

D. Determine $s(20)$. Use what you observed in parts A to C to simplify the expression so that your answer uses the smallest possible radical.

mixed radical

a radical with coefficient other than 1; for example, $2\sqrt{3}$

Reflecting

E. Determine $s(20)$ as a decimal. Why would the decimal answer for $\sqrt{24}$ not be considered exact?

F. To express $\sqrt{24}$ as a mixed radical, explain why using the factors $\sqrt{4} \times \sqrt{6}$ is a better choice than using $\sqrt{3} \times \sqrt{8}$.

G. If a and b are positive whole numbers, describe how $\sqrt{ab}$ is related to $\sqrt{a} \times \sqrt{b}$.

H. If $a > 0$, why is $b\sqrt{a}$ a simpler form of $\sqrt{ab^2}$?

APPLY the Math

EXAMPLE 1 Simplifying radicals by using a strategy involving perfect-square factors

Express each of the following as a mixed radical in lowest terms.

a) $\sqrt{72}$ **b)** $5\sqrt{27}$

Jasmine's Solution

a) $\sqrt{72} = \sqrt{36} \times \sqrt{2}$
$= 6\sqrt{2}$

I needed to find a perfect square number that divides evenly into 72. I could have chosen 4 or 9, but to put the mixed radical in lowest terms, I had to choose the greatest perfect square, which was 36. Once I expressed $\sqrt{72}$ as $\sqrt{36} \times \sqrt{2}$, I evaluated $\sqrt{36}$.

b) $5\sqrt{27} = 5 \times \sqrt{9} \times \sqrt{3}$
$= 5 \times 3 \times \sqrt{3}$
$= 15\sqrt{3}$

I found the largest perfect square that would divide evenly into 27; it was 9. I evaluated the square root of 9 and multiplied it by the coefficient 5.

EXAMPLE 2 Changing mixed radicals to entire radicals

Express each of the following as entire radicals.

a) $4\sqrt{5}$ **b)** $-6\sqrt{3}$

Sami's Solution

a) $4\sqrt{5} = 4 \times \sqrt{5}$
$= \sqrt{16} \times \sqrt{5}$
$= \sqrt{80}$

To create an entire radical, I had to change 4 into a square root. I expressed 4 as the square root of 16. Then I was able to multiply the numbers under the radical signs.

b) $-6\sqrt{3} = (-6) \times \sqrt{3}$
$= (-1) \times 6 \times \sqrt{3}$
$= (-1) \times \sqrt{36} \times \sqrt{3}$
$= -\sqrt{108}$

I knew that the negative sign would not go under the radical, since squares of real numbers are always positive. So I wrote -6 as the product of -1 and 6. I expressed 6 as $\sqrt{36}$ so that I could multiply the radical parts together to make an entire radical.

EXAMPLE 3 Multiplying radicals

Simplify.

a) $\sqrt{5} \times \sqrt{11}$ b) $-4\sqrt{6} \times 2\sqrt{6}$

Caleb's Solution

a) $\sqrt{5} \times \sqrt{11} = \sqrt{55}$

I multiplied the numbers under the radical signs together. 55 was not divisible by a perfect square, so my answer was in lowest terms.

b) $-4\sqrt{6} \times 2\sqrt{6} = (-4) \times 2 \times \sqrt{6} \times \sqrt{6}$
$= (-8)\sqrt{36}$

A mixed radical is the product of the integer and the radical, so I grouped together the integer products and the radical products.

$= (-8) \times 6$
$= -48$

Since 36 is a perfect square, I was able to simplify.

EXAMPLE 4 Adding radicals

In Don's research, he may also have to add expressions that contain radicals. Can he add radicals that are **like radicals**? What about other radicals?

like radicals

radicals that have the same number under the radical symbol, such as $3\sqrt{6}$ and $-2\sqrt{6}$

Marta's Solution

$\sqrt{3} \doteq 1.732$
$\sqrt{5} \doteq 2.236$
$\sqrt{3} + \sqrt{5} = 1.732 + 2.236$
$= 3.968$
$\sqrt{8} \doteq 2.828$
So $\sqrt{3} + \sqrt{5} \neq \sqrt{8}.$

I used my calculator to evaluate two radicals that were not like each other: $\sqrt{3}$ and $\sqrt{5}$. I rounded each value to 3 decimal places and then performed the addition. When I calculated $\sqrt{8}$, I found that it was not equal to $\sqrt{3} + \sqrt{5}$.

It looks like I cannot add radicals together if the numbers under the radical signs are different.

$3\sqrt{2} = \sqrt{2} + \sqrt{2} + \sqrt{2}$
$3\sqrt{2} + \sqrt{2} = \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2}$
$= 4\sqrt{2}$

Also,

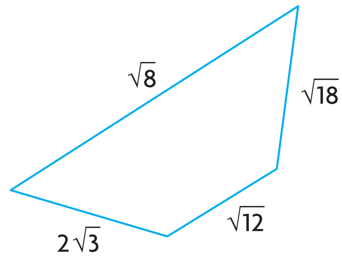
$3\sqrt{2} + \sqrt{2} = \sqrt{2}(3 + 1)$
$= \sqrt{2} \times 4$
$= 4\sqrt{2}$

Then I tried two like radicals. I used $3\sqrt{2}$ and $\sqrt{2}$. I expressed $3\sqrt{2}$ as the sum of three $\sqrt{2}$s. When I added $\sqrt{2}$ to this sum, I had 4 of them altogether, or $4\sqrt{2}$.

It makes sense that I can add like radicals by adding the integers in front of the radicals together.

EXAMPLE 5 Solving a problem involving radicals

Calculate the perimeter. Leave your answer in simplest radical form.



Robert's Solution

$$\begin{aligned} P &= \sqrt{8} + 2\sqrt{3} + \sqrt{12} + \sqrt{18} \\ &= \sqrt{4} \times \sqrt{2} + 2\sqrt{3} + \sqrt{4} \times \sqrt{3} + \sqrt{9} \times \sqrt{2} \\ &= 2\sqrt{2} + 2\sqrt{3} + 2\sqrt{3} + 3\sqrt{2} \\ &= 2\sqrt{2} + 3\sqrt{2} + 2\sqrt{3} + 2\sqrt{3} \\ &= 5\sqrt{2} + 4\sqrt{3} \end{aligned}$$

To find the perimeter I needed to add up the sides. I can add only like radicals. I factored the numbers I could by using perfect squares to see if any of these are like radicals.

I grouped, and then added the like radicals together.

EXAMPLE 6 Multiplying binomial radical expressions

Simplify $\left(3 - \sqrt{6}\right)\left(2 + \sqrt{24}\right)$.

Barak's Solution

$$\left(3 - \sqrt{6}\right)\left(2 + \sqrt{24}\right)$$

I simplified this expression by first expanding the quantities in brackets.

$$\begin{aligned} &= 6 + 3\sqrt{24} - 2\sqrt{6} - \sqrt{144} \\ &= 6 + 3(\sqrt{4} \times \sqrt{6}) - 2\sqrt{6} - 12 \\ &= 6 + 3(2\sqrt{6}) - 2\sqrt{6} - 12 \\ &= 6 - 12 + 6\sqrt{6} - 2\sqrt{6} \end{aligned}$$

After I multiplied the terms, I noticed that some of them could be simplified further. I factored $\sqrt{24}$ by using a perfect square. I then evaluated $\sqrt{144}$ and simplified.

$$= -6 + 4\sqrt{6}$$

I collected and combined like radicals.

In Summary

Key Idea

- Entire radicals can sometimes be simplified by expressing them as the product of two radicals, one of which contains a perfect square. This results in a mixed radical.
  - $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$ for $a \geq 0, b \geq 0$
  - $c\sqrt{a} \times d\sqrt{b} = cd\sqrt{ab}$ for $a \geq 0, b \geq 0$

Need to Know

- The only radicals that can be added or subtracted into a single term are like radicals.
- An answer containing a radical is an exact answer. An answer containing a decimal is an approximate answer.
- A mixed radical is in simplest form when the smallest possible number is written under the radical sign.

CHECK YOUR Understanding

1. Express each of these as mixed radicals in simplest form.

a) $\sqrt{27}$
b) $\sqrt{50}$
c) $\sqrt{98}$
d) $\sqrt{32}$

2. Simplify.

a) $\sqrt{5} \times \sqrt{7}$
b) $\sqrt{11} \times \sqrt{6}$
c) $2\sqrt{3} \times 5\sqrt{2}$
d) $-4\sqrt{3} \times 8\sqrt{13}$

3. Simplify.

a) $4\sqrt{5} + 3\sqrt{5}$
b) $9\sqrt{7} - 4\sqrt{7}$
c) $3\sqrt{3} + 8\sqrt{2} - 4\sqrt{3} + 11\sqrt{2}$
d) $\sqrt{8} - \sqrt{18}$

PRACTISING

4. Express as a mixed radical in simplest form.

a) $3\sqrt{12}$
b) $-5\sqrt{125}$
c) $10\sqrt{40}$
d) $-\frac{1}{2}\sqrt{60}$
e) $\frac{2}{3}\sqrt{45}$
f) $-\frac{9}{10}\sqrt{1200}$

5. Simplify.

a) $\sqrt{3}(2 - \sqrt{5})$
b) $2\sqrt{2}(\sqrt{7} + 3\sqrt{3})$
c) $(4\sqrt{2})^2$
d) $(-2\sqrt{3})^3$
e) $4\sqrt{3} \times 3\sqrt{6}$
f) $-7\sqrt{2} \times 5\sqrt{8}$

6. Simplify.
   a) $\sqrt{8} - \sqrt{32}$
   b) $\sqrt{12} + \sqrt{18} - \sqrt{27} + \sqrt{50}$
   c) $3\sqrt{98} - 5\sqrt{72}$
   d) $-4\sqrt{200} + 5\sqrt{242}$
   e) $-5\sqrt{45} + \sqrt{52} + 3\sqrt{125}$
   f) $7\sqrt{12} - 3\sqrt{28} + \frac{1}{2}\sqrt{48} + \frac{2}{3}\sqrt{63}$

7. Simplify.
   K a) $(6 - \sqrt{5})(3 + 2\sqrt{10})$
   b) $(2 + 3\sqrt{3})^2$
   c) $(\sqrt{2} + \sqrt{5})(\sqrt{2} - \sqrt{5})$
   d) $(3\sqrt{3} + 4\sqrt{2})(\sqrt{3} - 2\sqrt{2})$
   e) $(2\sqrt{5} - 3\sqrt{7})^2$
   f) $(1 - \sqrt{3})(2 + \sqrt{6})(5 + \sqrt{2})$

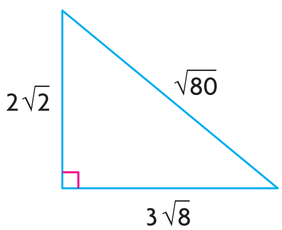
For questions 8 to 12, calculate the exact values and express your answers is simplest radical form.

8. Calculate the length of the diagonal of a square with side length 4 cm.

9. A square has an area of 450 cm^2. Calculate the side length.

10. Determine the length of the diagonal of a rectangle with dimensions 3 cm × 9 cm.

11. Determine the length of the line segment from $A(-2, 7)$ to $B(4, 1)$.
   A
12. Calculate the perimeter and area of this triangle.



13. If $a > 0$ and $b > 0$, which is greatest, $\left(\sqrt{a} + \sqrt{b}\right)^2$ or $\sqrt{a^2} + \sqrt{b^2}$?
   T
14. Give three mixed radicals that are equivalent to $\sqrt{200}$. Which answer is in
   C simplest radical form? Explain how you know.

Extending

15. Express each radical in simplest radical form.
   a) $\sqrt{a^3}$
   b) $\sqrt{x^5y^6}$
   c) $5\sqrt{n^7} - 2n\sqrt{n^5}$
   d) $(\sqrt{p} + 2\sqrt{q})(\sqrt{q} - \sqrt{p})$

16. Simplify $\sqrt{\sqrt{\sqrt{4096}}}$.

17. Solve $(\sqrt{2})^x = 256$ for x.