3.3 The Inverse of a Quadratic Function

GOAL

Determine the inverse of a quadratic function, given different representations.

YOU WILL NEED

- graph paper
- ruler
- graphing calculator

INVESTIGATE the Math

Suzanne needs to make a box in the shape of a cube. She has 864 cm^2 of cardboard to use. She wants to use all of the material provided.

? How long will each side of Suzanne's box be?

A. Copy and complete this table.

Cube Side Length (cm)	1	2	3	4	5	6	7	8	9	10
Area of Each Face (cm^2)	1	4								
Surface Area (cm^2)	6	24								

B. Draw a graph of surface area versus side length. What type of function is this? Explain how you know.

C. Determine the equation that represents the cube's surface area as a function of its side length. Use function notation and state the domain and range.

D. How would you calculate the inverse of this function to describe the side length of the cube if you know its surface area?

E. Make a table of values for the inverse of the surface area function.

F. Draw a graph of the inverse. Compare the graph of the inverse with the original graph. Is the inverse a function? Explain.

G. State the domain and range of the inverse.

H. Write the equation that represents the cube's side length for a given surface area.

I. Use your equation from part H to determine the largest cube Suzanne will be able to construct.

Reflecting

J. How are the surface area function and its inverse related
- **a)** in the table of values?
- **b)** in their graphs?
- **c)** in their domains and ranges?

K. How is any quadratic function related to its inverse
  a) in their domains and ranges?
  b) in their equations?

APPLY the Math

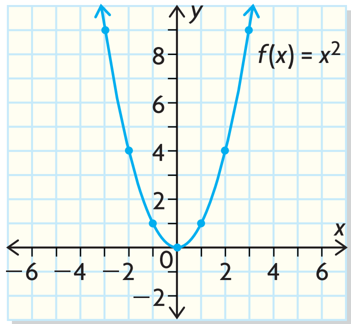
EXAMPLE 1 Determining the domain and range of the inverse of a quadratic function

Given the graph of $f(x) = x^2$,

a) graph the inverse relation

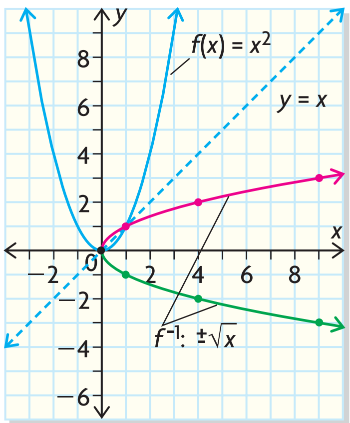
b) state the domain and range of $f(x) = x^2$ and the inverse relation

c) determine whether the inverse of $f(x) = x^2$ is also a function. Give a reason for your answer.



Paul's Solution

a)



To graph the inverse of $f(x) = x^2$, I took the coordinates of each point on the original graph and switched the x- and y-coordinates. For example, (2, 4) became (4, 2). I had to do this because the input value becomes the output value in the inverse, and vice versa.

The graph of the inverse is a reflection of the original function about the line $y = x$.

b) The domain of $f(x) = x^2$ is $\{x \in \mathbf{R}\}$. The range of $f(x) = x^2$ is $\{y \in \mathbf{R} \mid y \geq 0\}$. Therefore, the domain of f^{-1} is $\{x \in \mathbf{R} \mid x \geq 0\}$, and the range is $\{y \in \mathbf{R}\}$.

The domain of the inverse is the range of the original function. The range of the inverse is the domain of the original function.

c) The inverse of $f(x) = x^2$ is not a function.

I knew this because the inverse fails the vertical-line test: For each number in the domain except 0, there are two values in the range.

EXAMPLE 2 Determining the equation of the inverse of a quadratic function

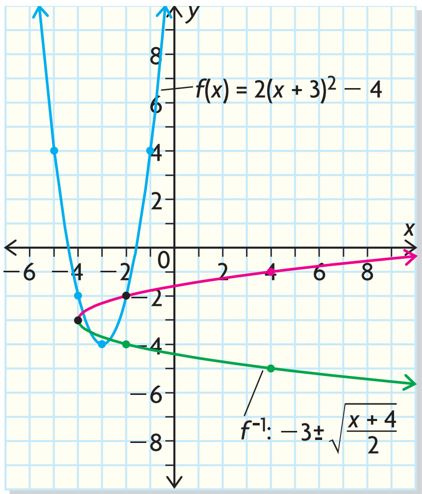
Given the quadratic function $f(x) = 2(x + 3)^2 - 4$,

a) graph $f(x)$ and its inverse

b) determine the equation of the inverse

Prashant's Solution

a)



I graphed $f(x)$ by plotting the vertex, $(-3, -4)$. The parabola opens up because the value of a is positive. I found $f(-2) = -2$ and $f(-1) = 4$, which are also the same values of $f(-4)$ and $f(-5)$, respectively.

To graph the inverse, I interchanged the x- and y-coordinates of the points on the original function.

b)

$$f(x) = 2(x + 3)^2 - 4$$
$$y = 2(x + 3)^2 - 4$$
$$x = 2(y + 3)^2 - 4$$

First I wrote the equation with y replacing $f(x)$, because y represents the output value in the function. To find the equation of the inverse, I interchanged x and y in the original function.

$$x + 4 = 2(y + 3)^2$$
$$\frac{x + 4}{2} = (y + 3)^2$$
$$\pm\sqrt{\frac{x + 4}{2}} = y + 3$$
$$-3 \pm \sqrt{\frac{x + 4}{2}} = y$$

I then rearranged the equation and solved for y by using the inverse of the operations given in the original function. I could tell from the graph of the inverse that there were two parts to the inverse function, an upper branch and a lower branch. The upper branch came from taking the positive square root of both sides, the lower from taking the negative square root.

For $f(x)$ restricted to $x \geq -3$,

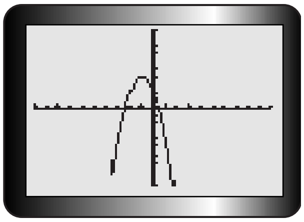
$$f^{-1}(x) = -3 + \sqrt{\frac{x + 4}{2}}$$

I couldn't write $f^{-1}(x)$ for y, since the inverse is not a function. But if I restricted the original domain to $x \geq -3$, then I would use only one branch of the inverse, and I could write it in function notation.

EXAMPLE 3 Using a graphing calculator as a strategy to graph a quadratic function and its inverse

Using a graphing calculator, graph $f(x) = -2(x + 1)^2 + 4$ and its inverse.

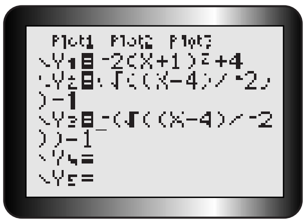
Bonnie's Solution



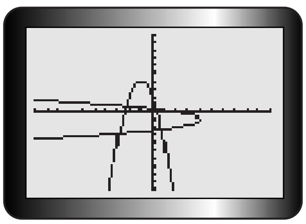
I entered the function into the equation editor at **Y1**. I used **WINDOW** settings $-10 \leq x \leq 10$, $-10 \leq y \leq 10$, because I knew the vertex was located at $(-1, 4)$ and the parabola opened down.

$$f(x) = -2(x + 1)^2 + 4$$
$$y = -2(x + 1)^2 + 4$$
$$x = -2(y + 1)^2 + 4$$
$$x - 4 = -2(y + 1)^2$$
$$\frac{x - 4}{-2} = (y + 1)^2$$
$$\pm\sqrt{\frac{x - 4}{-2}} = y + 1$$
$$\pm\sqrt{\frac{x - 4}{-2}} - 1 = y$$

To graph the inverse, I needed to find the equation of the inverse. I switched x and y and used inverse operations with the original function.



I entered both parts of the inverse separately. I entered $y = \sqrt{\frac{x - 4}{-2}} - 1$ into Y2 and $y = -\sqrt{\frac{x - 4}{-2}} - 1$ into Y3.



The calculator displayed the function and its inverse.

EXAMPLE 4 Solving a problem by using the inverse of a quadratic function

The rate of change in the surface area of a cell culture can be modelled by the function $S(t) = -0.005(t-6)^2 + 0.18$, where $S(t)$ is the rate of change in the surface area in square millimetres per hour at time t in hours, and $0 \le t \le 12$.

a) State the domain and range of $S(t)$.

b) Determine the model that describes time in terms of the surface area.

c) Determine the domain and range of the new model.

Thomas' Solution

a) Domain $= \{t \in \mathbf{R} \mid 0 \le t \le 12\}$

The domain is given in the problem as part of the model.

Range $= \{S \in \mathbf{R} \mid 0 \le S \le 0.18\}$

This function is a parabola that opens down. The vertex is (6, 0.18), so the maximum value is 0.18. The surface area also cannot be negative, so 0 is the minimum value.

$$S = -0.005(t-6)^2 + 0.18$$

To find the inverse of the original function, I solved the given equation for t by using the inverse operations.

$$S - 0.18 = -0.005(t-6)^2$$

$$\frac{S - 0.18}{-0.005} = (t-6)^2$$

$$\pm\sqrt{\frac{S - 0.18}{-0.005}} = t - 6$$

I did not interchange S and t in this case because S always means surface area and t always means time.

$$t = 6 \pm \sqrt{\frac{S - 0.18}{-0.005}}$$

$$t = 6 \pm \sqrt{\frac{-S + 0.18}{0.005}}$$

The domain and range of the new model:
Domain $= \{S \in \mathbf{R} \mid 0 \le S \le 0.18\}$

The value under the square root sign has to be positive, so the greatest value S can have is 0.18. For values greater than 0.18, the numerator would be positive, so the value under the square root would be negative.

Surface area cannot be less than zero, so S must be at least 0.

Range $= \{t \in \mathbf{R} \mid 0 \le t \le 12\}$

If $S = 0.18$, then the value of t is 6. If $S = 0$, then $t = 6 \pm 6$, so $t = 0$ or $t = 12$.

Therefore, the range values are between 0 and 12.

In Summary

Key Ideas

- The inverse of the original function undoes what the original function has done. It can be used to determine which values of the original dependent variable produce given values for the original independent variable.
- The inverse of a quadratic function defined over all the real numbers is not a function. It is a parabolic relation that opens either to the left or to the right. If the original quadratic opens up $(a > 0)$, the inverse opens to the right. If the original quadratic opens down $(a < 0)$, the inverse opens to the left.

Need to Know

- The equation of the inverse of a quadratic can be found by interchanging *x* and *y* in vertex form and solving for *y*.
- In the equation of the inverse of a quadratic function, the positive square root function represents the upper branch of the parabola, while the negative root represents the lower branch.
- The inverse of a quadratic function can be a function if the domain of the original function is restricted.

CHECK *Your Understanding*

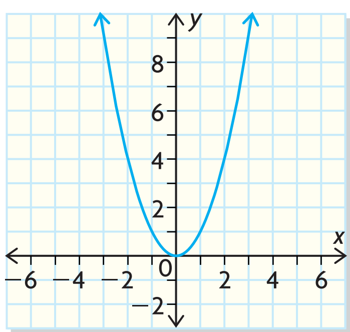
1. Each set of ordered pairs defines a parabola. Graph the relation and its inverse.

a) $\{(0, 0), (1, 3), (2, 12), (3, 27)\}$

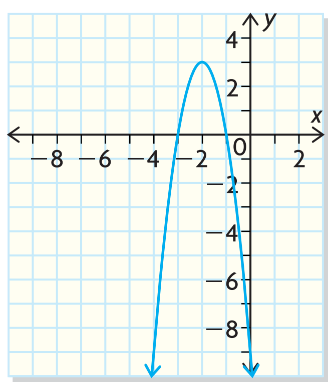
b) $\{(-3, -4), (-2, 1), (-1, 4), (0, 5), (1, 4), (2, 1), (3, -4)\}$

2. Given the graph of $f(x)$, graph the inverse relation.

a)



b)



3. Given $f(x) = 2x^2 - 1$, determine the equation of the inverse.

PRACTISING

4. Given $f(x) = 7 - 2(x - 1)^2, x \geq 1$, determine
   a) $f(3)$ b) $f^{-1}(x)$ c) $f^{-1}(5)$ d) $f^{-1}(2a + 7)$

5. a) Sketch the graph of $f(x) = 3(x - 2)^2 - 2$.
   b) Sketch the graph of its inverse on the same axes.

6. a) Graph $g(x) = -\sqrt{x}$ for $x \geq 0$.
   b) Graph its inverse on the same axes.
   c) State the domain and range of $g^{-1}(x)$.
   d) Determine the equation for $g^{-1}(x)$.

7. **K** Given $f(x) = -(x + 1)^2 - 3$ for $x \geq -1$, determine the equation for $f^{-1}(x)$. Graph the function and its inverse on the same axes.

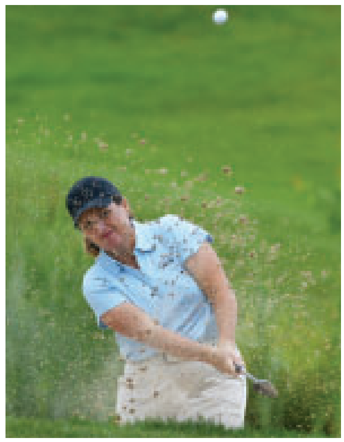
8. Given $f(x) = \frac{1}{2}(x - 5)^2 + 3$, find the equation for $f^{-1}(x)$ for the part of the function where $x \leq 5$. Use a graphing calculator to graph $f^{-1}(x)$.

9. For $-2 < x < 3$ and $f(x) = 3x^2 - 6x$, determine
   a) the domain and range of $f(x)$
   b) the equation of $f^{-1}(x)$ if $f(x)$ is further restricted to $1 < x < 3$

10. **A** The height of a ball thrown from a balcony can be modelled by the function $h(t) = -5t^2 + 10t + 35$, where $h(t)$ is the height above the ground, in metres, at time t seconds after it is thrown.
    a) Write $h(t)$ in vertex form.
    b) Determine the domain and range of $h(t)$.
    c) Determine the model that describes time in terms of the height.
    d) What are the domain and range of the new model?

11. The height of a golf ball after Lori Kane hits it is shown in the table.

Time (s)	0	0.5	1	1.5	2	2.5
Height (m)	0	12.375	22.5	30.375	36.0	39.375

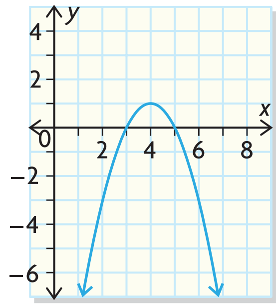


    a) Use first and second differences to extend the table.
    b) Graph the data and a curve of good fit for the relationship.
    c) Graph the inverse relation and its curve of good fit.
    d) Is the inverse a function? Explain.

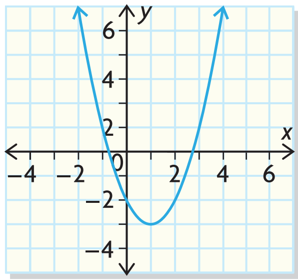
12. **T** Consider $f(x) = -2x^2 + 3x - 1$.
    a) Determine the vertex of the parabola.
    b) Graph $f(x)$.
    c) Graph $f^{-1}(x)$ for $y \geq 0.75$.
    d) Determine the domain and range of $f^{-1}(x)$ for $y \geq 0.75$.
    e) Why were the values of x restricted in parts (c) and (d)?

13. Each graph shows a function f that is a parabola or a branch of a parabola.

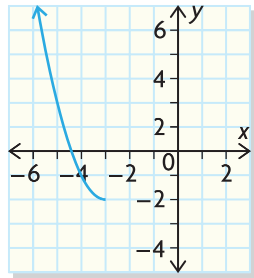
a)



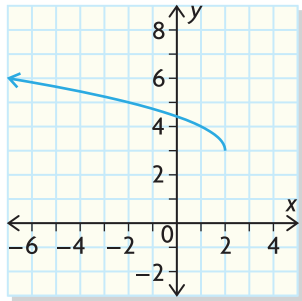
b)



c)



d)



i) Determine $f(x)$.

ii) Graph f^{-1}.

iii) State restrictions on the domain or range of f to make its inverse a function.

iv) Determine the equation(s) for f^{-1}.

14. What must happen for the inverse of a quadratic function defined over all the real numbers also to be a function?

15. C **a)** If you are given a quadratic function in standard form, explain how you could determine the equation of its inverse.

b) If the domain of the quadratic function is $\{x \in \mathbf{R}\}$, will its inverse be a function? Explain.

Extending

16. A meat department manager discovers that she can sell $m(x)$ kilograms of ground beef in a week, where $m(x) = 14\,700 - 3040x$, if she sells it at x dollars per kilogram. She pays her supplier \$3.21/kg for the beef.

a) Determine an algebraic expression for $P(x)$, where $P(x)$ represents the total profit in dollars for 1 week.

b) Find the equation for the inverse relation. Interpret its meaning.

c) Write an expression in function notation to represent the price that will earn \$1900 in profit. Evaluate and explain.

d) Determine the price that will maximize profit.

e) The supply cost drops to \$3.10/kg. What price should the manager set? How much profit will be earned at this price?

17. You are given the relation $x = 4 - 4y + y^2$.

a) Graph the relation.

b) Determine the domain and range of the relation.

c) Determine the equation of the inverse.

d) Is the inverse a function? Explain.