3.2 Determining Maximum and Minimum Values of a Quadratic Function

GOAL

Use a variety of strategies to determine the maximum or minimum value of a quadratic function.

LEARN ABOUT the Math

A golfer attempts to hit a golf ball over a gorge from a platform above the ground. The function that models the height of the ball is $h(t) = -5t^2 + 40t + 100$, where $h(t)$ is the height in metres at time t seconds after contact. There are power lines 185 m above the ground.

? Will the golf ball hit the power lines?

EXAMPLE 1 Selecting a strategy to find the vertex

Using the function for the golf ball's height, determine whether the ball will hit the power line.

Jonah's Solution: Completing the Square

$h(t) = -5t^2 + 40t + 100$

I needed to find the maximum height of the ball to compare it to the height of the power lines.

a is negative. The graph of *h*(*t*) is a parabola that opens down. Its maximum value occurs at the vertex.

$h(t) = -5(t^2 - 8t) + 100$

I put the function into vertex form by **completing the square**.

I factored -5 from the t^2 and t terms.

$= -5(t^2 - 8t + 16 - 16) + 100$

I divided the coefficient of t in half, then squared it to create a perfect-square trinomial.

By adding 16, I changed the value of the expression. To make up for this, I subtracted 16.

$= -5(t^2 - 8t + 16) + 80 + 100$ ← I grouped the first 3 terms that formed the perfect square and moved the subtracted value of 16 outside the brackets by multiplying by -5.

$= -5(t - 4)^2 + 180$ ← I factored the perfect square and collected like terms.

The vertex is (4, 180). The maximum height will be 180 m after 4 s.

Since the power lines are 185 m above the ground, the ball will not hit them. ← Since the vertex is at the maximum height, the ball goes up only 180 m.

Sophia's Solution: Factoring to Determine the Zeros

$h(t) = -5t^2 + 40t + 100$ ← The maximum height of the golf ball is at the vertex of the parabola.

The vertex is located on the axis of symmetry, which is always in the middle of the two zeros of the function. To find the zeros, I factored the quadratic.

$h(t) = -5(t^2 - 8t - 20)$

$h(t) = -5(t - 10)(t + 2)$ ← I divided -5 out as a common factor. Inside the brackets was a simple trinomial I could factor.

$0 = -5(t - 10)(t + 2)$

$t = 10$ or $t = -2$ ← The zeros are the values that make $h(t) = 0$. I found them by setting each factor equal to 0 and solving the resulting equations.

For the axis of symmetry, ← I added the zeros and divided the result by 2 to locate the axis of symmetry. This was also the *x*-coordinate, or in this case, the *t*-coordinate of the vertex.

$$t = \frac{10 + (-2)}{2}$$

$$t = \frac{8}{2}$$

$$t = 4$$

The *t*-coordinate of the vertex is 4.

$h(4) = -5(4 - 10)(4 + 2)$ ← To find the *y*-value, or height *h*, I substituted $t = 4$ into the factored form of the equation. Alternatively, I could have substituted into the function in standard form.

$= -5(-6)(6)$

$= 180$

The vertex is (4, 180). The maximum height will be 180 m, after 4 s. Since the power lines are 185 m above the ground, the ball will not hit them.

Reflecting

A. How can you tell from the algebraic form of a quadratic function whether the function has a maximum or a minimum value?

B. Compare the two methods for determining the vertex of a quadratic function. How are they the same? How are they different?

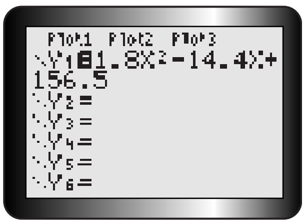
C. Not all quadratic functions have zeros. Which method allows you to find the vertex without finding the zeros? Explain.

APPLY the Math

EXAMPLE 2 — Using the graphing calculator as a strategy to determine the minimum value

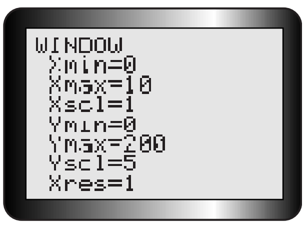
The cost, $c(x)$, in dollars per hour of running a certain steamboat is modelled by the quadratic function $c(x) = 1.8x^2 - 14.4x + 156.5$, where x is the speed in kilometres per hour. At what speed should the boat travel to achieve the minimum cost?

Rita's Solution



This parabola opens up. Therefore, the minimum value will be at the vertex.

I used a graphing calculator because the numbers in the question were decimals and it would not have been easy to complete the square or factor.

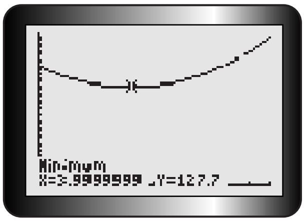


I chose a **WINDOW** that would display the graph. I chose both the *x*- and *y*-values to have a minimum of 0, since neither cost nor speed could be negative.

I picked a maximum *x* of 10 and an *x* scale of 1. I estimated the corresponding maximum *y* from the function.

Tech Support

For help using the graphing calculator to determine the minimum value of a function, see Technical Appendix, B-9.



I used the minimum operation to locate the vertex.

The minimum cost to operate the steamboat is \$127.70/h, when the boat is travelling at about 4 km/h.

The vertex is (4, 127.70).

EXAMPLE 3 Solving a problem to determine when the maximum value occurs

The demand function for a new magazine is $p(x) = -6x + 40$, where $p(x)$ represents the selling price, in thousands of dollars, of the magazine and x is the number sold, in thousands. The cost function is $C(x) = 4x + 48$. Calculate the maximum profit and the number of magazines sold that will produce the maximum profit.

Levi's Solution

$$\text{Revenue} = \text{Demand} \times \text{Number sold}$$
$$= [p(x)](x)$$

I found the revenue function by multiplying the demand function by the number of magazines sold.

$$\text{Profit} = \text{Revenue} - \text{Cost}$$

To find the profit function, I subtracted the cost function from the revenue function and simplified.

$$P(x) = [p(x)](x) - C(x)$$
$$= (-6x + 40)(x) - (4x + 48)$$
$$= -6x^2 + 40x - 4x - 48$$
$$= -6x^2 + 36x - 48$$

The coefficient of x^2 is negative, so the parabola opens down with its maximum value at the vertex. Instead of completing the square, I determined two points symmetrically opposite each other.

$$P(x) = -6x(x - 6) - 48$$

I started by factoring the common factor $-6x$ from $-6x^2$ and $36x$.

Communication *Tip*

The demand function $p(x)$ is the relation between the price of an item and the number of items sold, x. The cost function $C(x)$ is the total cost of making x items. Revenue is the money brought in by selling x items. Revenue is the product of the demand function and the number sold. Profit is the difference between revenue and cost.

$-6x(x - 6) = 0$

$x = 0 \quad \text{or} \quad x = 6$

Points on the graph of the profit function are $(0, -48)$ and $(6, -48)$.

I knew that the x-intercepts of the graph of $y = -6x(x - 6)$ would help me find the two points I needed on the graph of the profit function, since both functions have the same axis of symmetry.

The axis of symmetry is $x = \frac{0 + 6}{2}$ or $x = 3$. So the x-coordinate of the vertex is 3.

I found the axis of symmetry, which gave me the x-coordinate of the vertex.

$$P(3) = -6(3)^2 + 36(3) - 48$$
$$= -54 + 108 - 48$$
$$= 6$$

I substituted $x = 3$ into the function to determine the profit. I remembered that x is in *thousands* of magazines sold, and $P(x)$ is in *thousands* of dollars.

The maximum profit is \$6000, when 3000 magazines are sold.

In Summary

Key Idea

- The maximum or minimum value of a quadratic function is the y-coordinate of the vertex.

Need to Know

- If $a > 0$ in standard form, factored form, or vertex form, then the parabola opens up. The quadratic has a minimum value.
- If $a < 0$ in standard form, factored form, or vertex form, then the parabola opens down. The quadratic has a maximum value.
- The vertex can be found from the standard form $f(x) = ax^2 + bx + c$ algebraically in several ways:
  - by completing the square to put the quadratic in vertex form $f(x) = a(x - h)^2 + k$
  - by expressing the quadratic in factored form $f(x) = a(x - r)(x - s)$, if possible, and averaging the zeros at r and s to locate the axis of symmetry. This will give the x-coordinate of the vertex
  - by factoring out the common factor from $ax^2 + bx$ to determine two points on the parabola that are symmetrically opposite each other, and averaging the x-coordinates of these points to determine the x-coordinate of the vertex
  - by using a graphing calculator

CHECK Your Understanding

1. Which of the following quadratic functions will have a maximum value? Explain how you know.

a) $y = -x^2 + 7x$

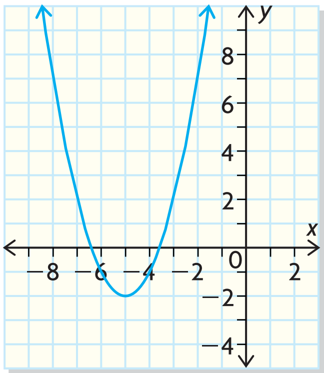
b) $f(x) = 3(x - 1)^2 - 4$

c) $f(x) = -4(x + 2)(x - 3)$

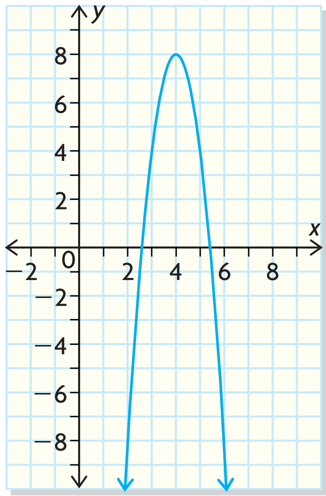
d) $g(x) = 4x^2 + 3x - 5$

2. State the vertex of each parabola and indicate the maximum or minimum value of the function.

a)



b)



3. Determine the maximum or minimum value for each.

a) $y = -4(x + 1)^2 + 6$

b) $f(x) = (x - 5)^2$

c) $f(x) = -2x(x - 4)$

d) $g(x) = 2x^2 - 7$

PRACTISING

4. Determine the maximum or minimum value. Use at least two different methods.
K

a) $y = x^2 - 4x - 1$

b) $f(x) = x^2 - 8x + 12$

c) $y = 2x^2 + 12x$

d) $y = -3x^2 - 12x + 15$

e) $y = 3x(x - 2) + 5$

f) $g(x) = -2(x + 1)^2 - 5$

5. Each function is the demand function of some item, where x is the number of items sold, in thousands. Determine

i) the revenue function

ii) the maximum revenue in thousands of dollars

a) $p(x) = -x + 5$

b) $p(x) = -4x + 12$

c) $p(x) = -0.6x + 15$

d) $p(x) = -1.2x + 4.8$

6. Use a graphing calculator to determine the maximum or minimum value. Round to two decimal places where necessary.

a) $f(x) = 2x^2 - 6.5x + 3.2$

b) $f(x) = -3.6x^2 + 4.8x$

7. For each pair of revenue and cost functions, determine

i) the profit function

ii) the value of x that maximizes profit

a) $R(x) = -x^2 + 24x$, $C(x) = 12x + 28$

b) $R(x) = -2x^2 + 32x$, $C(x) = 14x + 45$

c) $R(x) = -3x^2 + 26x$, $C(x) = 8x + 18$

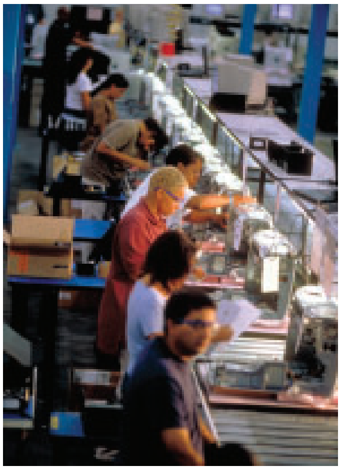
d) $R(x) = -2x^2 + 25x$, $C(x) = 3x + 17$

8. The height of a ball thrown vertically upward from a rooftop is modelled by $h(t) = -5t^2 + 20t + 50$, where $h(t)$ is the ball's height above the ground, in metres, at time t seconds after the throw.

a) Determine the maximum height of the ball.

b) How long does it take for the ball to reach its maximum height?

c) How high is the rooftop?



9. The cost function in a computer manufacturing plant is $C(x) = 0.28x^2 - 0.7x + 1$, where $C(x)$ is the cost per hour in millions of dollars and x is the number of items produced per hour in thousands. Determine the minimum production cost.

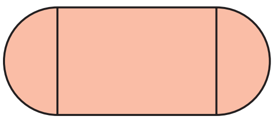
10. Show that the value of $3x^2 - 6x + 5$ cannot be less than 1.

11. **A** The profit $P(x)$ of a cosmetics company, in thousands of dollars, is given by $P(x) = -5x^2 + 400x - 2550$, where x is the amount spent on advertising, in thousands of dollars.

a) Determine the maximum profit the company can make.

b) Determine the amount spent on advertising that will result in the maximum profit.

c) What amount must be spent on advertising to obtain a profit of at least \$4 000 000?



12. **T** A high school is planning to build a new playing field surrounded by a running track. The track coach wants two laps around the track to be 1000 m. The football coach wants the rectangular infield area to be as large as possible. Can both coaches be satisfied? Explain your answer.

13. **C** Compare the methods for finding the minimum value of the quadratic function $f(x) = 3x^2 - 7x + 2$. Which method would you choose for this particular function? Give a reason for your answer.

Extending

14. A rock is thrown straight up in the air from an initial height h_0, in metres, with initial velocity v_0, in metres per second. The height in metres above the ground after t seconds is given by $h(t) = -4.9t^2 + v_0t + h_0$. Find an expression for the time it takes the rock to reach its maximum height.

15. A ticket to a school dance is \$8. Usually, 300 students attend. The dance committee knows that for every \$1 increase in the price of a ticket, 30 fewer students attend the dance. What ticket price will maximize the revenue?