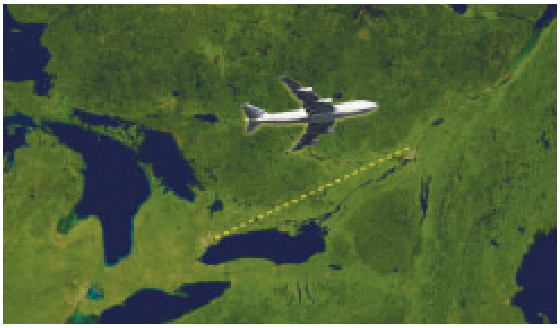
2.7 Adding and Subtracting Rational Expressions

GOAL

Develop strategies for adding and subtracting rational expressions.

LEARN ABOUT the Math



A jet flies along a straight path from Toronto to Montreal and back again. The straight-line distance between these cities is 540 km. On Monday, the jet made the round trip when there was no wind. On Friday, it made the round trip when there was a constant wind blowing from Toronto to Montreal at 80 km/h. While travelling in still air, the jet travels at constant speed.

? Which round trip takes less time?

EXAMPLE 1 Selecting a strategy for adding and subtracting rational expressions

Write expressions for the length of time required to fly from Toronto to Montreal in each situation. Determine which trip takes less time.

Basil's Solution

v is the jet's airspeed in still air.

$v + 80$ is the jet's airspeed from Toronto to Montreal.

$v - 80$ is the jet's airspeed from Montreal to Toronto.

I assigned a variable, v, to the jet's airspeed in still air, since its value is not given. So the speed with the wind from Toronto and the speed against the wind from Montreal are $v + 80$ and $v - 80$, respectively.

$\frac{540}{v}$ is the time elapsed when there is no wind.

$\frac{540}{v + 80}$ is the time elapsed from Toronto to Montreal.

$\frac{540}{v - 80}$ is the time elapsed from Montreal to Toronto.

Using the relation time $= \frac{\text{distance}}{\text{speed}}$, I determined expressions for the elapsed time for each way of the trip at each airspeed.

No wind

$$T_1 = \frac{540}{v} + \frac{540}{v}$$
$$= \frac{1080}{v}$$

I let T_1 represent the time on Monday, with no wind.

I let T_2 represent the time on Friday, with wind.

I found the round-trip times by adding the times for each way.

Wind

$$T_2 = \frac{540}{v + 80} + \frac{540}{v - 80}$$
$$= \frac{540(v - 80) + 540(v + 80)}{(v + 80)(v - 80)}$$
$$= \frac{1080v}{v^2 - 6400}$$

$$T_1 = \frac{1080}{v} \times \frac{v}{v}$$

I noticed that T_1 has the denominator v while T_2's denominator contains v^2. To compare T_1 with T_2, I need to have the same denominator, so I rewrote T_1 by multiplying its numerator and denominator by v.

Now the numerators are both the same.

$$= \frac{1080v}{v^2}$$

T_2 has a smaller denominator because 6400 is subtracted from v^2. Since I am dealing with division, the lesser of the two expressions is the one with the greater denominator, in this case T_1.

The trip without wind took less time.

Reflecting

A. Why were the expressions for time rational expressions?

B. How can you determine a common denominator of two rational functions?

C. How do the methods for adding and subtracting rational expressions compare with those for adding and subtracting rational numbers?

APPLY the Math

EXAMPLE 2 Using the lowest common denominator strategy to add rational expressions

Simplify and state any restrictions on the variables: $\frac{3}{8x^2} + \frac{1}{4x} - \frac{5}{6x^3}$.

Sheila's Solution

$\text{LCD} = 24x^3$

I found the **lowest common denominator** (LCD) by finding the least common multiple of $8x^2$, $4x$, and $6x^3$.

$$\frac{3}{8x^2} + \frac{1}{4x} - \frac{5}{6x^3}$$
$$= \frac{(3x)3}{(3x)8x^3} + \frac{(6x^2)1}{(6x^2)4x} - \frac{(4)5}{(4)6x^3}$$

I used the LCD to rewrite each term. For each term, I multiplied the denominator by the factor necessary to get the LCD. Then, I multiplied the numerator by the same factor.

$$= \frac{9x + 6x^2 - 20}{24x^3};\ x \neq 0$$

I added and subtracted the numerators.

I determined the restrictions on the denominator by solving $24x^3 = 0$.

EXAMPLE 3 Using a factoring strategy to add expressions with binomial denominators

Simplify and state any restrictions on the variables: $\frac{3n}{2n + 1} + \frac{4}{n - 3}$.

Tom's Solution

$\text{LCD} = (2n + 1)(n - 3)$

I found the lowest common denominator by multiplying both denominators.

$$\frac{3n}{2n + 1} + \frac{4}{n - 3}$$
$$= \frac{(n - 3)3n}{(2n + 1)(n - 3)} + \frac{(2n + 1)4}{(2n + 1)(n - 3)}$$

I used the lowest common denominator to rewrite each term.

$$= \frac{(n-3)3n + (2n+1)4}{(2n+1)(n-3)}$$

$$= \frac{3n^2 - 9n + 8n + 4}{(2n+1)(n-3)}$$

I simplified by expanding the numerators.

$$= \frac{3n^2 - n + 4}{(2n+1)(n-3)};\ x \neq -\frac{1}{2}, 3$$

I collected like terms and determined the restrictions by solving $(2n + 1)(n - 3) = 0$.

EXAMPLE 4 Using a factoring strategy to add expressions with quadratic denominators

Simplify and state any restrictions on the variables: $\frac{2t}{t^2 - 1} - \frac{t+2}{t^2 + 3t - 4}$.

Frank's Solution

$$\frac{2t}{t^2 - 1} - \frac{t+2}{t^2 + 3t - 4}$$

I factored the denominators.

To find the LCD, I created a product by using the three unique factors:

$(t - 1)(t + 1)(t + 4)$

$$= \frac{2t}{(t-1)(t+1)} - \frac{t+2}{(t+4)(t-1)}$$

$$= \frac{(t+4)2t}{(t-1)(t+1)(t+4)} - \frac{(t+1)(t+2)}{(t+1)(t-1)(t+4)}$$

I used the lowest common denominator to rewrite each term.

$$= \frac{2t^2 + 8t - t^2 - 2t - t - 2}{(t-1)(t+1)(t+4)}$$

I simplified by expanding the numerators.

$$= \frac{t^2 + 5t - 2}{(t-1)(t+1)(t+4)};\ t \neq 1, -1, -4$$

I collected like terms and determined the restrictions by solving $(t - 1)(t + 1)(t + 4) = 0$.

In Summary

Key Idea

- The procedures for adding or subtracting rational functions are the same as those for adding and subtracting rational numbers. When rational expressions are added or subtracted, they must have a common denominator.

Need to Know

- To add or subtract rational functions or expressions, determine the lowest common denominator (LCD). To do this, factor all the denominators. The LCD consists of the product of any common factors and all the unique factors.
- The LCD is not always the product of all the denominators.
- After finding the LCD, rewrite each term using the LCD as the denominator and then add or subtract numerators.
- Restrictions are found by finding the zeros of all denominators, that is, the zeros of the LCD.

CHECK *Your Understanding*

1. Simplify. State any restrictions on the variables.

a) $\frac{1}{3} + \frac{5}{4}$

b) $\frac{2x}{5} + \frac{6x}{2}$

c) $\frac{5}{4x^2} + \frac{1}{7x^3}$

d) $\frac{2}{x} + \frac{6}{x^2}$

2. Simplify. State any restrictions on the variables.

a) $\frac{5}{9} - \frac{2}{3}$

b) $\frac{5y}{3} - \frac{y}{2}$

c) $\frac{5}{3x^2} - \frac{7}{5}$

d) $\frac{6}{3xy} - \frac{5}{y^2}$

3. Simplify. State any restrictions on the variables.

a) $\frac{3}{x - 3} - \frac{7}{5x - 1}$

b) $\frac{2}{x + 3} + \frac{7}{x^2 - 9}$

c) $\frac{5}{x^2 - 4x + 3} - \frac{9}{x^2 - 2x + 1}$

4. a) Evaluate $\frac{2}{(x^2 - 9)} + \frac{3}{(x - 3)}$ when $x = 5$.

b) Simplify the original expression by adding.

c) Evaluate the simplified expression when $x = 5$. What do you notice?

PRACTISING

5. Simplify. State any restrictions on the variables.

a) $\frac{2x}{3} + \frac{3x}{4} - \frac{x}{6}$

b) $\frac{3}{t^4} + \frac{1}{2t^2} - \frac{3}{5t}$

c) $\frac{2x}{3y} - \frac{x^2}{4y^3} + \frac{3}{5y^4}$

d) $\frac{n}{m} + \frac{m}{n} - m$

6. Simplify. State any restrictions on the variables.

a) $\frac{7}{a - 4} + \frac{2}{a}$

b) $\frac{4}{3x - 2} + 6$

c) $\frac{5}{x + 4} + \frac{7}{x + 3}$

d) $\frac{6}{2n - 3} - \frac{4}{n - 5}$

e) $\frac{7x}{x + 4} + \frac{3x}{x - 6}$

f) $\frac{7}{2x - 6} + \frac{4}{10x - 15}$

7. Simplify. State any restrictions on the variables.

a) $\dfrac{3}{x + 1} + \dfrac{4}{x^2 - 3x - 4}$

b) $\dfrac{2t}{t - 4} - \dfrac{5t}{t^2 - 16}$

c) $\dfrac{3}{t^2 + t - 6} + \dfrac{5}{(t + 3)^2}$

d) $\dfrac{4x}{x^2 + 6x + 8} - \dfrac{3x}{x^2 - 3x - 10}$

e) $\dfrac{x - 1}{x^2 - 9} + \dfrac{x + 7}{x^2 - 5x + 6}$

f) $\dfrac{2t + 1}{2t^2 - 14t + 24} + \dfrac{5t}{4t^2 - 8t - 12}$

8. Simplify. State any restrictions on the variables.

a) $\dfrac{3}{4x^2 + 7x + 3} - \dfrac{5}{16x^2 + 24x + 9}$

b) $\dfrac{a - 1}{a^2 - 8a + 15} - \dfrac{a - 2}{2a^2 - 9a - 5}$

c) $\dfrac{3x + 2}{4x^2 - 1} + \dfrac{2x - 5}{4x^2 + 4x + 1}$

9. Simplify. State any restrictions on the variables. Remember the order of operations.

a) $\dfrac{2x^3}{3y^2} \times \dfrac{9y}{10x} - \dfrac{2y}{3x}$

b) $\dfrac{x + 1}{2x - 6} \div \dfrac{2(x + 1)^2}{2 - x} + \dfrac{11}{x - 2}$

c) $\dfrac{p + 1}{p^2 + 2p - 35} + \dfrac{p^2 + p - 12}{p^2 - 2p - 24} \times \dfrac{p^2 - 4p - 12}{p^2 + 2p - 15}$

d) $\dfrac{5m - n}{2m + n} - \dfrac{4m^2 - 4mn + n^2}{4m^2 - n^2} \div \dfrac{6m^2 - mn - n^2}{3m + 15n}$

10. **K** Simplify. State any restrictions on the variables.

a) $\dfrac{3m + 2}{2} + \dfrac{4m + 5}{5}$

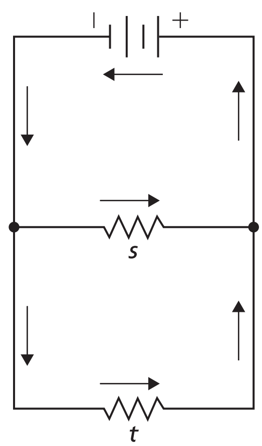
b) $\dfrac{5}{x^2} - \dfrac{3}{4x^3}$

c) $\dfrac{2}{y + 1} - \dfrac{3}{y - 2}$

d) $\dfrac{2x}{x^2 + x - 6} + \dfrac{5}{x^2 + 2x - 8}$

11. **T** When two resistors, s and t, are connected in parallel, their combined resistance, R, is given by $\dfrac{1}{R} = \dfrac{1}{s} + \dfrac{1}{t}$.

If s is increased by 1 unit and t is decreased by 1 unit, what is the change in R?



12. Fred drove his car a distance of $2x$ km in 3 h. Later, he drove a distance of $x + 100$ km in 2 h. Use the equation speed $= \frac{\text{distance}}{\text{time}}$.
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a) Write a simplified expression for the difference between the first speed and the second speed.

b) Determine the values of x for which the speed was greater for the second trip.

13. Matthew is attending a very loud concert by The Discarded. To avoid permanent ear damage, he decides to move farther from the stage. Sound intensity is given by the formula $I = \frac{k}{d^2}$, where k is a constant and d is the distance in metres from the listener to the source of the sound.

Determine an expression for the decrease in sound intensity if Matthew moves x metres farther from the stage.

14. **a)** For two rational numbers in simplified form, the lowest common denominator is always one of the following:
C

- **i)** one of the denominators
- **ii)** the product of the denominators
- **iii)** none of the above

Give an example of each of these.

b) Explain how you would determine the LCD of two simplified rational functions with different quadratic denominators. Illustrate with examples.

Extending

15. In question 11, you encountered an equation of the form $\frac{1}{x} + \frac{1}{y} = \frac{1}{z}$, which can be written as $\frac{1}{x} = \frac{1}{z} - \frac{1}{y}$. Suppose you want to determine natural-number solutions of this equation; for example, $\frac{1}{6} = \frac{1}{2} - \frac{1}{3}$ and $\frac{1}{20} = \frac{1}{4} - \frac{1}{5}$.

a) Show that the difference between reciprocals of consecutive positive integers is the reciprocal of their product,

$$\frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$$

b) State two more solutions of the equation $\frac{1}{x} = \frac{1}{z} - \frac{1}{y}$.

16. A Pythagorean triple is a triple of natural numbers satisfying the equation $a^2 + b^2 = c^2$. One way to produce a Pythagorean triple is to add the reciprocals of any two consecutive even or odd numbers. For example, $\frac{1}{5} + \frac{1}{7} = \frac{12}{35}$.

Now, $12^2 + 35^2 = 1369$. This is a triple if 1369 is a square, which it is: $1369 = 37^2$. So 12, 35, 37 is a triple.

a) Show that this method always produces a triple.

b) Determine a triple using the method.