2.6 Multiplying and Dividing Rational Expressions

GOAL

Develop strategies for multiplying and dividing rational expressions.

LEARN ABOUT the Math

Tulia is telling Daisy about something that her chemistry teacher was demonstrating. It is about the variables X, Y, and Z.

Tulia: I didn't catch how X, Y, and Z are related.
Daisy: Tell me what the units of the three quantities are.
Tulia: They were $\frac{\text{mol}}{\text{L} \cdot \text{s}}$, $\frac{\text{L}}{\text{mol}}$, and s^{-1}, respectively.
Daisy: I have no idea what any of those mean.
Tulia: I guess I'll have to look for help elsewhere.
Daisy: Hold on. All of the units are like rational expressions, so maybe there is some operation that relates them.
Tulia: Like what?

? How are the three quantities related?

EXAMPLE 1 Selecting a strategy for multiplying rational expressions

Use multiplication to show how the expressions $\frac{\text{mol}}{\text{L} \cdot \text{s}}$, $\frac{\text{L}}{\text{mol}}$, and s^{-1} are related.

Luke's Solution

$$\frac{\overset{1}{\cancel{\text{mol}}}}{\underset{1}{\cancel{\text{L}}} \cdot \text{s}} \times \frac{\overset{1}{\cancel{\text{L}}}}{\underset{1}{\cancel{\text{mol}}}} = \frac{1}{\text{s}}$$

Daisy suggested that the quantities are related by multiplication. I wrote the quantities as a product and then simplified the expression.

$$\text{s}^{-1} = \frac{1}{\text{s}}$$

$$\text{So, } \frac{\text{mol}}{\text{L} \cdot \text{s}} \times \frac{\text{L}}{\text{mol}} = \text{s}^{-1}$$

I wrote the variable with a negative exponent as a rational expression with a positive exponent. This showed that the quantities are related by multiplication.

Reflecting

A. Was Daisy correct in saying that the units of *X*, *Y*, and *Z* were rational expressions?

B. Explain why Daisy's method for multiplying the rational expressions was correct.

APPLY the Math

EXAMPLE 2 Selecting a strategy for multiplying simple rational expressions

Simplify and state the restrictions.

$$\frac{6x^2}{5xy} \times \frac{15xy^3}{8xy^4}$$

Buzz's Solution

$$\frac{6x^2}{5xy} \times \frac{15xy^3}{8xy^4}$$

When I substituted values for the variables, the result was a fraction, so I multiplied the rational expressions the same way as when I multiply fractions. For $x = 1$, $y = 1$ the expression becomes

$$\frac{6(1)^2}{5(1)(1)} \times \frac{15(1)(1)^3}{8(1)(1)^4} = \frac{6}{5} \times \frac{15}{8} = \frac{9}{4}$$

$$= \frac{90x^3y^3}{40x^2y^5}$$

I multiplied the numerators and then the denominators. I did this by multiplying the coefficients and adding the exponents when the base was the same.

$$= \frac{\overset{1}{\cancel{10x^2y^3}}(9x)}{\underset{1}{\cancel{10x^2y^3}}(4y^2)}$$

I factored the numerator and denominator by dividing out the GCF $10x^2y^3$. Then I divided both the numerator and denominator by the GCF.

$$= \frac{9x}{4y^2};\ x \neq 0,\ y \neq 0$$

I determined the restrictions by setting the original denominator to zero: $40x^2y^5 = 0$. So, neither x nor y can be zero.

EXAMPLE 3 Selecting a strategy for multiplying more complex rational expressions

Simplify and state the restrictions.

$$\frac{x^2 - 4}{(x + 6)^2} \times \frac{x^2 + 9x + 18}{2(2 - x)}$$

Willy's Solution

$$\frac{x^2 - 4}{(x + 6)^2} \times \frac{x^2 + 9x + 18}{2(2 - x)}$$

$$= \frac{(x - 2)(x + 2)}{(x + 6)^2} \times \frac{(x + 3)(x + 6)}{2(2 - x)}$$

I factored the numerators and denominators.

$$= \frac{(x - 2)(x + 2)}{(x + 6)^2} \times \frac{(x + 3)(x + 6)}{-2(-2 + x)}$$

I noticed that $(x - 2)$ in the numerator was the opposite of $(2 - x)$ in the denominator. I divided out the common factor -1 from $(-2 + x)$ to get the signs the same in these factors.

$$= \frac{\overset{1}{\cancel{(x - 2)}}(x + 2)(x + 3)\cancel{(x + 6)}}{-2(x + 6)^{\cancel{2}\,1}\underset{1}{\cancel{(-2 + x)}}}$$

I multiplied the numerators and denominators and then divided out the common factors.

$$= \frac{-(x + 2)(x + 3)}{2(x + 6)};\ x \neq -6, 2$$

I determined the restrictions on the denominators by solving the equations $(x + 6)^2 = 0$ and $(-2 + x) = 0$.

EXAMPLE 4 Selecting a strategy for dividing rational expressions

Simplify and state the restrictions.

$$\frac{21p - 3p^2}{16p + 4p^2} \div \frac{14 - 9p + p^2}{12 + 7p + p^2}$$

Aurora's Solution

$$\frac{21p - 3p^2}{16p + 4p^2} \div \frac{14 - 9p + p^2}{12 + 7p + p^2}$$

$$= \frac{21p - 3p^2}{16p + 4p^2} \times \frac{12 + 7p + p^2}{14 - 9p + p^2}$$

When I substituted values for the variables, the result was a fraction, so I divided by multiplying the first rational expression by the reciprocal of the second, just as I would for fractions.

$$= \frac{3p(7 - p)}{4p(4 + p)} \times \frac{(3 + p)(4 + p)}{(7 - p)(2 - p)}$$

I factored the numerators and the denominators.

$$= \frac{3\cancel{p}\cancel{(7 - p)}^{1}}{4\cancel{p}\cancel{(4 + p)}_{1}} \times \frac{(3 + p)\cancel{(4 + p)}^{1}}{\cancel{(7 - p)}_{1}(2 - p)}$$

I simplified by dividing the numerators and denominators by all of their common factors.

$$= \frac{3(3 + p)}{4(2 - p)}; p \neq 0, -4, 7, 2, -3$$

I used the factored form of each denominator to determine the zeros by solving for p in $4p = 0$, $(4 + p) = 0$, $(7 - p) = 0$, $(2 - p) = 0$, and $(3 + p) = 0$.

In Summary

Key Idea

- The procedures you use to multiply or divide rational numbers can be used to multiply and divide rational expressions. That is, if A, B, C, and D are polynomials, then

$$\frac{A}{B} \times \frac{C}{D} = \frac{AC}{BD}, \text{ provided that } B, D \neq 0$$

$$\frac{A}{B} \div \frac{C}{D} = \frac{A}{B} \times \frac{D}{C} = \frac{AD}{BC}, \text{ provided that } B, D, \text{ and } C \neq 0$$

(*continued*)

Need to Know

- To multiply rational expressions,
  - factor the numerators and denominators, if possible
  - divide out any factors that are common to the numerator and denominator
  - multiply the numerators, multiply the denominators, and then write the result as a single rational expression
- To divide two rational expressions,
  - factor the numerators and denominators, if possible
  - multiply by the reciprocal of the divisor
  - divide out any factors common to the numerator and denominator
  - multiply the numerators and then multiply the denominators
  - write the result as a single rational expression
- To determine the restrictions, solve for the zeros of all of the denominators in the factored expression. In the case of division, both the numerator and denominator of the divisor must be used. Both are needed because the reciprocal of this expression is used in the calculation.

CHECK Your Understanding

1. Simplify. State any restrictions on the variables.

a) $\frac{2}{3} \times \frac{5}{8}$

b) $\frac{6x^2y}{5y^3} \times \frac{xy}{8}$

c) $\frac{(x + 1)(x - 5)}{(x + 4)} \times \frac{(x + 4)}{2(x - 5)}$

d) $\frac{x^2}{2x + 1} \times \frac{6x + 3}{5x}$

2. Simplify. State any restrictions on the variables.

a) $\frac{2x}{3} \div \frac{x^2}{5}$

b) $\frac{x - 7}{10} \div \frac{2x - 14}{25}$

c) $\frac{3x(x - 6)}{(x + 2)(x - 7)} \div \frac{(x - 6)}{(x + 2)}$

d) $\frac{x^2 - 1}{x - 2} \div \frac{x + 1}{12 - 6x}$

3. Simplify. State any restrictions on the variables.

a) $\frac{(x + 1)^2}{x^2 + 2x - 3} \times \frac{(x - 1)^2}{x^2 + 4x + 3}$

b) $\frac{2x + 10}{x^2 - 4x + 4} \div \frac{x^2 - 25}{x - 2}$

PRACTISING

4. Simplify. State any restrictions on the variables.

a) $\frac{2x^2}{7} \times \frac{21}{x}$

b) $\frac{7a}{3} \div \frac{14a^2}{5}$

c) $\frac{2x^3y}{3xy^2} \times \frac{9x}{4x^2y}$

d) $\frac{3a^2b^3}{2ab^2} \div \frac{9a^2b}{14a^2}$

5. Simplify. State any restrictions on the variables.

a) $\frac{2(x + 1)}{3} \times \frac{x - 1}{6(x + 1)}$

b) $\frac{3a - 6}{a + 2} \div \frac{a - 2}{a + 2}$

c) $\frac{2(x - 2)}{9x^3} \times \frac{12x^4}{2 - x}$

d) $\frac{3(m + 4)^2}{2m + 1} \div \frac{5(m + 4)}{7m + 14}$

6. Simplify. State any restrictions on the variables.

a) $\frac{(x + 1)(x - 3)}{(x + 2)^2} \times \frac{2(x + 2)}{(x - 3)(x + 3)}$

b) $\frac{2(n^2 - 7n + 12)}{n^2 - n - 6} \div \frac{5(n - 4)}{n^2 - 4}$

c) $\frac{2x^2 - x - 1}{x^2 - x - 6} \times \frac{6x^2 - 5x + 1}{8x^2 + 14x + 5}$

d) $\frac{9y^2 - 4}{4y - 12} \div \frac{9y^2 + 12y + 4}{18 - 6y}$

7. Simplify. State any restrictions on the variables.

a) $\frac{x^2 - 5xy + 4y^2}{x^2 + 3xy - 28y^2} \times \frac{x^2 + 2xy + y^2}{x^2 - y^2}$

b) $\frac{2a^2 - 12ab + 18b^2}{a^2 - 7ab + 10b^2} \div \frac{4a^2 - 12ab}{a^2 - 7ab + 10b^2}$

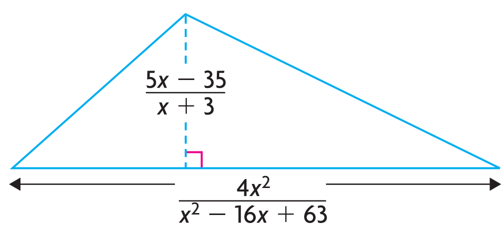
c) $\frac{10x^2 + 3xy - y^2}{9x^2 - y^2} \div \frac{6x^2 + 3xy}{12x + 4y}$

d) $\frac{15m^2 + mn - 2n^2}{2n - 14m} \times \frac{7m^2 - 8mn + n^2}{5m^2 + 7mn + 2n^2}$

8. **K** Simplify. State any restrictions on the variables.

$$\frac{x^2 + x - 6}{(2x - 1)^2} \times \frac{x(2x - 1)^2}{x^2 + 2x - 3} \div \frac{x^2 - 4}{3x}$$

9. **A** Determine the area of the triangle in simplified form. State the restrictions.



10. An object has mass $m = \dfrac{p + 1}{3p + 1}$ and density $\rho = \dfrac{p^2 - 1}{9p^2 + 6p + 1}$.

Determine its volume v, where $\rho = \dfrac{m}{v}$. State the restrictions on any variables.

11. **T** Liz claims that if $x = y$, she can show that $x + y = 0$ by following these steps:

Since $x = y$,

$x^2 = y^2$ ← I squared both sides of the equation.

So $x^2 - y^2 = 0$. ← I rearranged terms in the equation.

$\dfrac{x^2 - y^2}{x - y} = \dfrac{0}{x - y}$ ← I divided both sides by $x - y$.

$\dfrac{\overset{1}{\cancel{(x - y)}}(x + y)}{\underset{1}{\cancel{x - y}}} = \dfrac{0}{x - y}$ ← I factored and simplified.

$x + y = 0$

Sarit says that's impossible because if $x = 1$, then $y = 1$, since $x = y$. Substituting into Liz's final equation, $x + y = 0$, gives $1 + 1 = 2$, not 0.

Explain the error in Liz's reasoning.

12. **C**
a) Why do you usually factor all numerators and denominators *before* multiplying rational functions?
b) Are there any exceptions to the rule in part (a)? Explain.
c) Sam says that dividing two rational functions and multiplying the first function by the reciprocal of the second will produce the same function. Is this true? Explain.

Extending

13. Simplify. State any restrictions on the variables.

$$\frac{\dfrac{m^2 - mn}{6m^2 + 11mn + 3n^2} \div \dfrac{m^2 - n^2}{2m^2 - mn - 6n^2}}{\dfrac{4m^2 - 7mn - 2n^2}{3m^2 + 7mn + 2n^2}}$$

14. Newton's law of gravitation states that any two objects exert a gravitational force on each other due to their masses, $F_g = G\dfrac{m_1 m_2}{r^2}$, where F is the gravitational force, G is a constant (the universal gravitational constant), m_1 and m_2 are the masses of the objects, and r is the separation distance between the centres of objects. The mass of Mercury is 2.2 times greater than the mass of Pluto. Pluto is 102.1 times as far from the Sun as Mercury. How many times greater is the gravitational force between the Sun and Mercury than the gravitational force between the Sun and Pluto?