2.3 Factoring Polynomials

GOAL

Review and extend factoring skills.

LEARN ABOUT the Math

Mai claims that, for any natural number n, the function $f(n) = n^3 + 3n^2 + 2n + 6$ always generates values that are not **prime**.

? Is Mai's claim true?

EXAMPLE 1 Selecting a factoring strategy: Testing values of n to determine a pattern

Show that $f(n) = n^3 + 3n^2 + 2n + 6$ can be factored for any natural number, n.

Sally's Solution

$f(1) = 12 = 4 \times 3$

$f(2) = 30 = 5 \times 6$

$f(3) = 66 = 6 \times 11$

$f(4) = 126 = 7 \times 18$

$f(5) = 216 = 8 \times 27$

After some calculations and guess and check, I found a pattern. The first factor was of the form $n + 3$ and the second factor was of the form $n^2 + 2$.

$$f(n) = (n + 3)(n^2 + 2)$$
$$= n^3 + 3n^2 + 2n + 6$$

To confirm the pattern, I multiplied $(n + 3)$ by $(n^2 + 2)$.

Since both factors produce numbers greater than 1, $f(n)$ can never be expressed as the product of 1 and itself. So Mai's claim is true.

Sometimes an expression that doesn't appear to be factorable directly can be factored by grouping terms of the expression and dividing out common factors.

EXAMPLE 2 Selecting a factoring strategy: Grouping

Factor $f(n) = n^3 + 3n^2 + 2n + 6$ by grouping.

Noah's Solution

$$f(n) = n^3 + 3n^2 + 2n + 6$$

$$= (n^3 + 3n^2) + (2n + 6)$$

I separated $f(n)$ into two groups: the first two terms and the last two terms.

$$= n^2(n + 3) + 2(n + 3)$$

I factored each group by dividing by its common factor.

$$= (n + 3)(n^2 + 2)$$

Then I factored by dividing each term by the common factor $n + 3$.

Both factors produce numbers greater than 1, so $f(n)$ can never be expressed as the product of 1 and itself. So Mai's claim is true.

Reflecting

A. Why is Noah's method called factoring by grouping?

B. What are the advantages and disadvantages of Sally's and Noah's methods of factoring?

APPLY the Math

EXAMPLE 3 Selecting factoring strategies: Quadratic expressions

Factor.

a) $x^2 - x - 30$
b) $18x^2 - 50$
c) $10x^2 - x - 3$
d) $9x^2 + 30x + 25$
e) $2x^2 + x + 3$

Winnie's Solution

a) $x^2 - x - 30$

$$= (x + 5)(x - 6)$$

This is a trinomial of the form $ax^2 + bx + c$, where $a = 1$. I can factor it by finding two numbers whose sum is -1 and whose product is -30. These numbers are 5 and -6.

Check: $(x + 5)(x - 6)$

$= (x + 5)x - (x + 5)6$

$= x^2 + 5x - 6x - 30$

$= x^2 - x - 30$

I checked the answer by multiplying the two factors.

b) $18x^2 - 50$

$= 2(9x^2 - 25)$

First I divided each term by the common factor, 2. This left a difference of squares.

$= 2(3x + 5)(3x - 5)$

I took the square root of $9x^2$ and 25 to get $3x$ and 5, respectively.

c) $10x^2 - x - 3$

This is a trinomial of the form $ax^2 + bx + c$, where $a \neq 1$, and it has no common factor.

$= 10x^2 + 5x - 6x - 3$

I used decomposition by finding two numbers whose sum is -1 and whose product is $(10)(-3) = -30$. These numbers are 5 and -6. I used them to "decompose" the middle term.

$= 5x(2x + 1) - 3(2x + 1)$

I factored the group consisting of the first two terms and the group consisting of the last two terms by dividing each group by its common factor.

$= (2x + 1)(5x - 3)$

I divided out the common factor of $2x + 1$ from each term.

d) $9x^2 + 30x + 25$

$= (3x + 5)^2$

I noticed that the first and last terms are perfect squares. The square roots are $3x$ and 5, respectively. The middle term is double the product of the two square roots, $2(3x)(5) = 30x$. So this trinomial is a perfect square, namely, the square of a binomial.

e) $2x^2 + x + 3$

Trinomials of this form may be factored by decomposition. I tried to come up with two integers whose sum is 1 and whose product is 6. There were no such integers, so the trinomial cannot be factored.

EXAMPLE 4 Selecting a factoring strategy: Grouping

Factor $f(x) = x^3 + x^2 + x + 1$.

Fred's Solution

$$\begin{aligned} f(x) &= x^3 + x^2 + x + 1 \\ &= (x^3 + x^2) + (x + 1) \\ &= x^2(x + 1) + (x + 1) \end{aligned}$$

I grouped pairs of terms. Then I factored the **greatest common factor** (GCF) from each pair.

$$= (x + 1)(x^2 + 1)$$

Then I factored out the greatest common factor, $(x + 1)$, to complete the factoring.

EXAMPLE 5 Selecting a factoring strategy: Grouping as a difference of squares

Factor $g(x) = x^2 - 6x + 9 - 4y^2$.

Fran's Solution

$$\begin{aligned} g(x) &= x^2 - 6x + 9 - 4y^2 \\ &= (x - 3)^2 - (2y)^2 \end{aligned}$$

I recognized that the group consisting of the first three terms was the square of the binomial $x - 3$ and the last term was the square of $2y$.

$$= (x - 3 - 2y)(x - 3 + 2y)$$

I factored the resulting expression by using a difference of squares.

In Summary

Key Ideas

- Factoring a polynomial means writing it as a product. So factoring is the opposite of expanding.

factoring

$$x^2 + 3x - 4 = (x + 4)(x - 1)$$

expanding

(continued)

- If a polynomial has more than three terms, you may be able to factor it by grouping. This is only possible if the grouping of terms allows you to divide out the same common factor from each group.

Need to Know

- To factor a polynomial fully means that only 1 and -1 remain as common factors in the factored expression.
- To factor polynomials fully, you can use factoring strategies that include
  - dividing by the greatest common factor (GCF)
  - recognizing a factorable trinomial of the form $ax^2 + bx + c$, where $a = 1$
  - recognizing a factorable trinomial of the form $ax^2 + bx + c$, where $a \neq 1$
  - recognizing a polynomial that can be factored as a difference of squares: $a^2 - b^2 = (a + b)(a - b)$
  - recognizing a polynomial that can be factored as a perfect square: $a^2 + 2ab + b^2 = (a + b)^2$ and $a^2 - 2ab + b^2 = (a - b)^2$
  - factoring by grouping

CHECK Your Understanding

1. Factor.

a) $x^2 - 6x - 27$

b) $25x^2 - 49$

c) $4x^2 + 20x + 25$

d) $6x^2 - x - 2$

2. Each expression given can be factored by grouping. Describe how you would group the terms to factor each.

a) $ac + bc - ad - bd$

b) $x^2 + 2x + 1 - y^2$

c) $x^2 - y^2 - 10y - 25$

3. Factor.

a) $x^2 - 3x - 28$

b) $36x^2 - 25$

c) $9x^2 - 42x + 49$

d) $2x^2 - 7x - 15$

PRACTISING

4. Factor.

a) $4x^3 - 6x^2 + 2x$

b) $3x^3y^2 - 9x^2y^4 + 3xy^3$

c) $4a(a + 1) - 3(a + 1)$

d) $7x^2(x + 1) - x(x + 1) + 6(x + 1)$

e) $5x(2 - x) + 4x(2x - 5) - (3x - 4)$

f) $4t(t^2 + 4t + 2) - 2t(3t^2 - 6t + 17)$

5. Factor.

a) $x^2 - 5x - 14$

b) $x^2 + 4xy - 5y^2$

c) $6m^2 - 90m + 324$

d) $2y^2 + 5y - 7$

e) $8a^2 - 2ab - 21b^2$

f) $16x^2 + 76x + 90$

6. Factor.

a) $x^2 - 9$

b) $4n^2 - 49$

c) $x^8 - 1$

d) $9(y - 1)^2 - 25$

e) $3x^2 - 27(2 - x)^2$

f) $-p^2q^2 + 81$

7. Factor.

a) $ax + ay + bx + by$

b) $2ab + 2a - 3b - 3$

c) $x^3 + x^2 - x - 1$

d) $1 - x^2 + 6x - 9$

e) $a^2 - b^2 + 25 + 10a$

f) $2m^2 + 10m + 10n - 2n^2$

8. **K** Andrij claims that the following statement is true:
$x^3 - y^3 = (x - y)(x^2 + y^2)$
Is Andrij correct? Justify your decision.

9. Factor.

a) $2x(x - 3) + 7(3 - x)$

b) $xy + 6x + 5y + 30$

c) $x^3 - x^2 - 4x + 4$

d) $y^2 - 49 + 14x - x^2$

e) $6x^2 - 21x - 12x + 42$

f) $12m^3 - 14m^2 - 30m + 35$

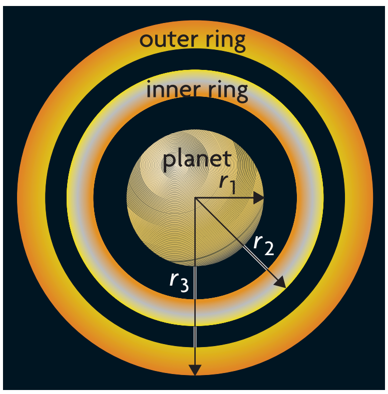
10. **T** Show that the function $f(n) = 2n^3 + n^2 + 6n + 3$ always produces a number that is divisible by an odd number greater than 1, for any natural number, n.

11. Sedna has designed a fishpond in the shape of a right triangle with two sides of length a and b and hypotenuse of length c.

a) Write an expression in factored form for a^2.

b) The hypotenuse is 3 m longer than b, and the sum of the lengths of the hypotenuse and b is 11 m. What are the lengths of the sides of the pond?

12. **A** Saturn is the ringed planet most people think of, but Uranus and Neptune also have rings. In addition, there are ringed planets outside our solar system. Consider the cross-section of the ringed planet shown.



a) Write factored expressions for

i) the area of the region between the planet and the inner ring

ii) the area of the region between the planet and the outer ring

iii) the difference of the areas from parts (i) and (ii)

b) What does the quantity in part (iii) represent?

13. Create a flow chart that will describe which strategies you would use to try
C to factor a polynomial. For each path through the flow chart, give an example of a polynomial that would follow that path and show its factored form. Explain how your flow chart could describe how to factor or show the non-factorability of any polynomial in this chapter.

Extending

14. The polynomial $x^4 - 5x^2 + 4$ is not factorable, but it can be factored by a form of completing the square:

$$\begin{aligned} & x^4 - 5x^2 + 4 \\ &= x^4 + 4x^2 + 4 - 4x^2 - 5x^2 \\ &= (x^2 + 2)^2 - 9x^2 \\ &= (x^2 + 2 - 3x)(x^2 + 2 + 3x) \\ &= (x^2 - 3x + 2)(x^2 + 3x + 2) \\ &= (x - 2)(x - 1)(x + 2)(x + 1) \end{aligned}$$

The first three terms form a perfect square.

This is now a difference of squares.

Use this strategy to factor each polynomial by creating a perfect square.

a) $x^4 + 3x^2 + 36$ **b)** $x^4 - 23x^2 + 49$

15. Expanding confirms that $x^2 - 1 = (x - 1)(x + 1)$ and also that $x^3 - 1 = (x - 1)(x^2 + x + 1)$.
Make conjectures and determine similar factorings for each expression.

a) $x^4 - 1$ **c)** $x^n - 1$
b) $x^5 - 1$ **d)** $x^n - y^n$

16. Mersenne numbers are numbers of the form $n = 2^m - 1$, where m is a natural number. For example, if $m = 6$, then $2^6 - 1 = 64 - 1 = 63$, and if $m = 5$, then $2^5 - 1 = 32 - 1 = 31$. $63 = 3 \times 21$ is a Mersenne number that is composite, and $31 = 1 \times 31$ is a Mersenne number that is prime. The French mathematician Mersenne was interested in finding the values of m that produced prime numbers, n.

a) $63 = 3(21)$ can also be expressed as $(2^2 - 1)(2^4 + 2^2 + 2^0)$, and $63 = 7(9)$ can also be expressed as $(2^3 - 1)(2^3 + 2^0)$. Expand the expressions that contain powers, treating them like polynomials, to show that you get $2^6 - 1$.

b) If $m = 9$, then $n = 2^9 - 1 = 511 = 7(73) = (2^3 - 1)(2^6 + 2^3 + 2^0)$. Using these types of patterns, show that $n = 2^{35} - 1$ is composite.

c) If m is composite, will the Mersenne number $n = 2^m - 1$ always be composite? Explain.