2.1 Adding and Subtracting Polynomials

YOU WILL NEED

- graphing calculator

Communication Tip

The numbers 1 and 2 in $a_1(t)$ and $a_2(t)$ are called subscripts. In this case, they are used to distinguish one function from the other. This distinction is necessary because both functions are named with the letter *a*.

GOAL

Determine whether polynomial expressions are equivalent.

LEARN ABOUT the Math

Fred enjoys working with model rockets. He wants to determine the difference in altitude of two different rockets when their fuel burns out and they begin to coast.

The altitudes, in metres, are given by these equations:

$$a_1(t) = -5t^2 + 100t + 1000$$

and

$$a_2(t) = -5t^2 + 75t + 1200$$

where t is the elapsed time, in seconds.

The difference in altitude, $f(t)$, is given by

$$f(t) = (-5t^2 + 100t + 1000) - (-5t^2 + 75t + 1200)$$

Fred simplified $f(t)$ to $g(t) = 175t + 2200$.

? Are the functions $f(t)$ and $g(t)$ equivalent?

EXAMPLE 1 Selecting a strategy to determine equivalence

Determine if $f(t)$ and $g(t)$ are equivalent functions.

Anita's Solution: Simplifying the Polynomial in $f(t)$

$$f(t) = (-5t^2 + 100t + 1000) - (-5t^2 + 75t + 1200)$$

$$= -5t^2 + 100t + 1000 + 5t^2 - 75t - 1200$$

Polynomials behave like numbers because, for any value of the variable, the result is a number. I know that with numbers, subtraction is equivalent to adding the opposite, so I subtracted the polynomials by adding the opposite of the second expression.

$$= 25t - 200$$

Then I collected like terms.

Since $g(t) \neq 25t - 200$, the functions are not equivalent.

If two expressions are not equivalent, then, for most values of t, their function values are different. The exception is when the functions intersect.

Maria's Solution: Evaluating the Functions for the Same Value of the Variable

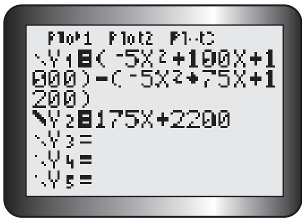
$$\begin{aligned} f(t) &= (-5t^2 + 100t + 1000) \\ &\quad - (-5t^2 + 75t + 1200) \\ f(0) &= (-5(0)^2 + 100(0) + 1000) \\ &\quad - (-5(0)^2 + 75(0) + 1200) \\ &= 1000 - 1200 \\ &= -200 \\ g(0) &= 175(0) + 2200 \\ &= 2200 \end{aligned}$$

I used $t = 0$ because it makes the calculations to find $f(0)$ and $g(0)$ easier.

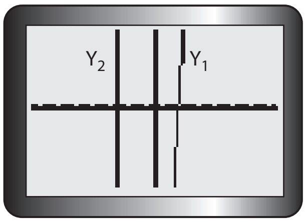
Since $f(0) \neq g(0)$, the functions are not equivalent.

If two functions are equivalent, then their graphs should be identical.

Sam's Solution: Graphing Both Functions



I graphed Fred's original function, $f(t)$, in Y_1 and his final function, $g(t)$, in Y_2. I zoomed out until I could see both functions.



Since the graphs are different, the functions must be different.

The functions are not equivalent.

Reflecting

A. How is subtracting two polynomials like subtracting integers? How is it different?

B. If Fred had not made an error when he simplified, whose method would have shown that his original and final expressions are equivalent?

C. What are the advantages and disadvantages of the three methods used to determine whether two polynomials are equivalent?

APPLY the Math

EXAMPLE 2 Reasoning whether two polynomials are equivalent

Nigel and Petra are hosting a dinner for 300 guests. Cheers banquet hall has quoted these charges:

- \$500, plus \$10 per person, for food,
- \$200, plus \$20 per person, for drinks, and
- a discount of \$5 per person if the number of guests exceeds 200.

Nigel and Petra have created two different functions for the total cost, where n represents the number of guests and $n > 200$.

Nigel's cost function: $C_1(n) = (10n + 500) + (20n + 200) - 5n$

Petra's cost function: $C_2(n) = (10n + 20n - 5n) + (500 + 200)$

Are the functions equivalent?

Lee's Solution

$$C_1(n) = (10n + 500) + (20n + 200) - 5n$$
$$= 10n + 20n - 5n + 500 + 200$$

Nigel's cost function was developed using the cost of each item.

The cost of the food is $(10n + 500)$. The cost of the drinks is $(20n + 200)$. The discount is $5n$.

$$= 25n + 700$$

I simplified by collecting like terms.

$$C_2(n) = (10n + 20n - 5n) + (500 + 200)$$
$$= 25n + 700$$

Petra's cost function was developed by adding the variable costs $(10n + 20n - 5n)$ and the fixed costs $(500 + 200)$. The result is two groups of like terms, which I then simplified.

$$C_1(n) = 25n + 700$$
$$C_2(n) = 25n + 700$$

Both cost functions simplify to the same function.

The two cost functions are equivalent.

EXAMPLE 3 Reasoning about the equivalence of expressions

Are the expressions $xy + xz + yz$ and $x^2 + y^2 + z^2$ equivalent?

Dwayne's Solution

To check for non-equivalence, I substituted some values for x, y, and z.

The values 0, 1, and -1 are often good choices, since they usually make expressions easy to evaluate.

$$xy + xz + yz = 0(0) + 0(1) + 0(1) = 0$$

I tried $x = 0$, $y = 0$, and $z = 1$ and evaluated the first expression.

$$x^2 + y^2 + z^2 = 0^2 + 0^2 + 1^2 = 1$$

I evaluated the second expression, using the same values for x, y, and z.

The expressions are not equivalent.

The expressions result in different values.

In Summary

Key Ideas

- Two polynomial functions or expressions are equivalent if
  - they simplify algebraically to give the same function or expression
  - they produce the same graph
- Two polynomial functions or expressions are not equivalent if
  - they result in different values when they are evaluated with the same numbers substituted for the variable(s)

Need to Know

- If you notice that two functions are equivalent at one value of a variable, it does not necessarily mean they are equivalent at *all* values of the variable. Evaluating both functions at a single value is sufficient to demonstrate non-equivalence, but it isn't enough to demonstrate equivalence.
- The sum of two or more polynomial functions or expressions can be determined by writing an expression for the sum of the polynomials and collecting like terms.
- The difference of two polynomial functions or expressions can be determined by adding the opposite of one polynomial and collecting like terms.

CHECK Your Understanding

1. Simplify.

a) $(3x^2 - 7x + 5) + (x^2 - x + 3)$
b) $(x^2 - 6x + 1) - (-x^2 - 6x + 5)$
c) $(2x^2 - 4x + 3) - (x^2 - 3x + 2) + (x^2 - 1)$

2. Show that $f(x)$ and $g(x)$ are equivalent by simplifying each.

$$f(x) = (2x - 1) - (3 - 4x) + (x + 2)$$
$$g(x) = (-x + 6) + (6x - 9) - (-2x - 1)$$

3. Show that $f(x)$ and $g(x)$ are not equivalent by evaluating each function at a suitable value of x.

$$f(x) = 2(x - 3) + 3(x - 3)$$
$$g(x) = 5(2x - 6)$$

PRACTISING

4. Simplify.

a) $(2a + 4c + 8) + (7a - 9c - 3)$
b) $(3x + 4y - 5z) + (2x^2 + 6z)$
c) $(6x + 2y + 9) + (-3x - 5y - 8)$
d) $(2x^2 - 7x + 6) + (x^2 - 2x - 9)$
e) $(-4x^2 - 2xy) + (6x^2 - 3xy + 2y^2)$
f) $(x^2 + y^2 + 8) + (4x^2 - 2y^2 - 9)$

5. Simplify.

a) $(m - n + 2p) - (3n + p - 7)$
b) $(-6m - 2q + 8) - (2m + 2q + 7)$
c) $(4a^2 - 9) - (a^3 + 2a - 9)$
d) $(2m^2 - 6mn + 8n^2) - (4m^2 - mn - 7n^2)$
e) $(3x^2 + 2y^2 + 7) - (4x^2 - 2y^2 - 8)$
f) $5x^2 - (2x^2 - 30) - (-20)$

6. Simplify.

a) $(2x - y) - (-3x + 4y) + (6x - 2y)$
b) $(3x^2 - 2x) + (x^2 - 7x) - (7x + 3)$
c) $(2x^2 + xy - y^2) - (x^2 - 4xy - y^2) + (3x^2 - 5xy)$
d) $(xy - xz + 4yz) + (2x - 3yz) - (4y - xz)$
e) $\left(\frac{1}{2}x + \frac{1}{3}y\right) - \left(\frac{1}{5}x - y\right)$
f) $\left(\frac{3}{4}x + \frac{1}{2}y\right) - \left(\frac{2}{3}x + \frac{1}{4}y - 1\right)$

7. Use two different methods to show that the expressions
K $(3x^2 - x) - (5x^2 - x)$ and $-2x^2 - 2x$ are not equivalent.

8. Determine whether each pair of functions is equivalent.

a) $f(x) = (2x^2 + 7x - 2) - (3x + 7)$ and
$g(x) = (x^2 + 12) + (x^2 + 4x - 17)$

b) $s_1(t) = (t + 2)^3$ and $s_2(t) = t^3 + 8$

c) $y_1 = (x - 1)(x)(x + 2)$ and $y_2 = 3x(x^2 - 1)$

d) $f(n) = 0.5n^2 + 2n - 3 + (1.5n^2 - 6)$ and
$g(n) = n^2 - n + 1 - (-n^2 - 3n + 10)$

e) $y_1 = 3p(q - 2) + 2p(q + 5)$ and $y_2 = p(q + 4)$

f) $f(m) = m(5 - m) - 2(2m - m^2)$ and
$g(m) = 4m^2(m - 1) - 3m^2 + 5m$

9. Determine two non-equivalent polynomials, $f(x)$ and $g(x)$, such that $f(0) = g(0)$ and $f(2) = g(2)$.

10. Kosuke wrote a mathematics contest consisting of 25 multiple-choice questions. The scoring system gave 6 points for a correct answer, 2 points for not answering a question, and 1 point for an incorrect answer. Kosuke got x correct answers and left y questions unanswered.

a) Write an expression for the number of questions he answered incorrectly.

b) Write an expression, in simplified form, for Kosuke's score.

c) Use the expressions you wrote in parts (a) and (b) to determine Kosuke's score if he answered 13 questions correctly and 7 incorrectly.

11. **A** The two equal sides of an isosceles triangle each have a length of $2x + 3y - 1$. The perimeter of the triangle is $7x + 9y$. Determine the length of the third side.

12. Tino owns a small company that produces and sells cellphone cases. The revenue and cost functions for Tino's company are shown below, where x represents the selling price in dollars.

$$\text{Revenue: } R(x) = -50x^2 + 2500x$$

$$\text{Cost: } C(x) = 150x + 9500$$

a) Write the simplified form of the profit function, $P(x) = R(x) - C(x)$.

b) What profit will the company make if it sells the cases for $12 each?

13. **T** For each pair of functions, label the pairs as equivalent, non-equivalent, or cannot be determined.

a) $f(2) = g(2)$

b) $h(3) = g(4)$

c) $j(8) \neq k(8)$

d) $l(5) \neq m(7)$

e) $n(x) = p(x)$ for all values of x in their domain

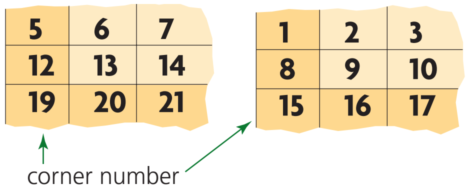
14. **C** Ramy used his graphing calculator to graph three different polynomial functions on the same axes. The equations of the functions all appeared to be different, but his calculator showed only two different graphs. He concluded that two of his functions were equivalent.

a) Is his conclusion correct? Explain.

b) How could he determine which, if any, of the functions were equivalent without using his graphing calculator?

Extending

15. Sanya noticed an interesting property of numbers that form a five-square capital-L pattern on a calendar page:



In each case that she checked, the sum of the five numbers was 18 less than five times the value of the number in the corner of the L. For example, in the calendars shown,

$$5 + 12 + 19 + 20 + 21 = 5(19) - 18$$

$$1 + 8 + 15 + 16 + 17 = 5(15) - 18$$

a) Show that this pattern holds for any numbers on the calendar page.

b) The sum of certain numbers in this pattern is 112. Determine the value of the corner number.

c) Write expressions for the sum of the five numbers, for the other three orientations of the L, when x is the corner number.

16. Since $70 = 5 \times 14$, 70 has 5 as a divisor. The number 70 can also be expressed as the sum of five consecutive natural numbers:

$$70 = 12 + 13 + 14 + 15 + 16$$

a) $105 = 5 \times 21$. Express 105 as the sum of five consecutive natural numbers.

b) Suppose m is a natural number that is greater than 2 and $n = 5m$. Express n as the sum of five consecutive natural numbers.

c) Express 91 as a sum of seven consecutive natural numbers.

17. a) Consider the linear functions $f(x) = ax + b$ and $g(x) = cx + d$. Suppose that $f(2) = g(2)$ and $f(5) = g(5)$. Show that the functions must be equivalent.

b) Consider the two quadratic functions $f(x) = ax^2 + bx + c$ and $g(x) = px^2 + qx + r$. Suppose that $f(2) = g(2)$, $f(3) = g(3)$, and $f(4) = g(4)$. Show that the functions must be equivalent.