1.7 Investigating Horizontal Stretches, Compressions, and Reflections

YOU WILL NEED

- graph paper (optional)
- graphing calculator

GOAL

Investigate and apply horizontal stretches, compressions, and reflections to parent functions

INVESTIGATE the Math

The function $p(L) = 2\pi\sqrt{\frac{1}{10}L}$ describes the time it takes a pendulum to complete one swing, from one side to the other and back, as a function of its length. In this formula,

- $p(L)$ represents the time in seconds
- L represents the pendulum's length in metres

Shannon wants to sketch the graph of this function. She knows that the parent function is $f(x) = \sqrt{x}$ and that the 2π causes a vertical stretch. She wonders what transformation is caused by multiplying x by $\frac{1}{10}$.

? What transformation must be applied to the graph of $y = f(x)$ to get the graph of $y = f(kx)$?

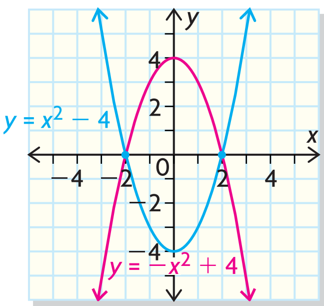
A. Copy and complete tables of values for $y = \sqrt{x}$ and $y = \sqrt{2x}$.

$y = \sqrt{x}$	
x	**y**
0	
1	
4	
9	
10	

$y = \sqrt{2x}$	
x	**y**
0	
0.5	
2	
4.5	
8	

Graph both functions on the same set of axes. State the domain and range of each function.

invariant point
a point on a graph (or figure) that is unchanged by a transformation—for example, (−2, 0) and (2, 0) for this graph and transformation



B. Compare the position and shape of the two graphs. Are there any **invariant points** on the graphs? Explain.

C. How could you transform the graph of $y = \sqrt{x}$ to obtain the graph of $y = \sqrt{2x}$?

D. Compare the points in the tables of values. How could you use the first table to obtain the second? What happens to the point (x, y) under this transformation? This transformation is called a *horizontal compression of factor* $\frac{1}{2}$. Explain why this is a good description.

E. Repeat parts A through D for $y = \sqrt{x}$ and $y = \sqrt{\frac{1}{2}x}$. What happens to the point (x, y) under this transformation? Describe the transformation in words.

F. Repeat parts A through D for $y = \sqrt{x}$ and $y = \sqrt{-x}$.

G. Using a graphing calculator, investigate the effect of varying k in $y = f(kx)$ on the graphs of the given parent functions. In each case, try values of k that are **i)** between 0 and 1, **ii)** greater than 1, and **iii)** less than 0.

a) $f(x) = x^2$ **b)** $f(x) = \frac{1}{x}$ **c)** $f(x) = |x|$

H. Write a summary of the results of your investigations. Explain how you would use the graph of $y = f(x)$ to sketch the graph of $y = f(kx)$.

Communication *Tip*

In describing vertical stretches/compressions $af(x)$, the scale factor is a, but for horizontal stretches/compressions $f(kx)$, the scale factor is $\frac{1}{k}$. In both cases, for a scale factor greater than 1, a stretch occurs, and for a scale factor between 0 and 1, a compression occurs.

Reflecting

I. What transformation is caused by multiplying L by $\frac{1}{10}$ in the pendulum function $p(L) = 2\pi\sqrt{\frac{1}{10}L}$?

J. How is the graph of $y = f(2x)$ different from the graph of $y = 2f(x)$?

K. How is the graph of $y = f(-x)$ different from the graph of $y = -f(x)$?

L. What effect does k in $y = f(kx)$ have on the graph of $y = f(x)$ when
i) $|k| > 1$? **ii)** $0 < |k| < 1$? **iii)** $k < 0$?

APPLY the Math

EXAMPLE 1 Applying horizontal stretches, compressions, and reflections

For each pair of functions, identify the parent function, describe the transformations required to graph them from the parent function, and sketch all three graphs on the same set of axes.

a) $y = (4x)^2$; $y = \left(\frac{1}{5}x\right)^2$ **b)** $y = |0.25x|$; $y = |-x - 3|$

Ana's Solution

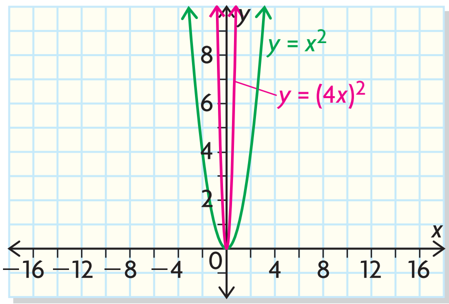
a) These functions are of the form $y = f(kx)^2$. The quadratic function $f(x) = x^2$ is the parent function.

I saw that these functions were $y = x^2$, with x multiplied by a number.

To graph $y = (4x)^2$, compress the graph of $y = (x)^2$ horizontally by the factor $\frac{1}{4}$.

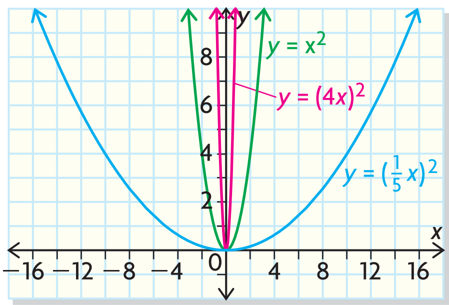
When x is multiplied by a number greater than 1, the graph is compressed horizontally. That makes sense, since the x-value required to make $y = 1$ is ± 1 for $y = x^2$, but is $\pm \frac{1}{4}$ for $y = (4x)^2$.

I multiplied the x-coordinates of the points (1, 1), (2, 4), and (3, 9) on $y = x^2$ by $\frac{1}{4}$ to find three points, $\left(\frac{1}{4}, 1\right)$, $\left(\frac{1}{2}, 4\right)$, and $\left(\frac{3}{4}, 9\right)$, on $y = (4x)^2$.



I plotted these points and joined them to the invariant point (0, 0) to graph one-half of the parabola. Then I used symmetry to complete the other half of the graph.

To graph $y = \left(\frac{1}{5}x^2\right)$, stretch the graph of $y = x^2$ horizontally by the factor 5.



This time, x is multiplied by a number between 0 and 1, so the graph is stretched horizontally. Instead of using an x-value of ± 1 to get a y-value of 1, I need an x-value of ± 5.

I used the same x-coordinates as before and multiplied by 5, which gave me points (5, 1), (10, 4), and (15, 9) to plot.

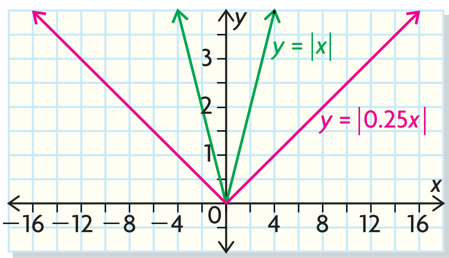
I used the invariant point (0, 0) and symmetry to complete the graph of $y = \left(\frac{1}{5}x\right)^2$.

b) The parent function is $f(x) = |x|$.

To graph $y = |0.25x|$, stretch the graph of $y = |x|$ horizontally by the factor 4.

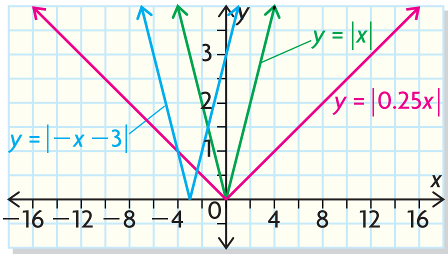
I knew from the absolute value signs that the parent function was the absolute value function.

I knew that the stretch factor was $\frac{1}{0.25} = 4$. The point that originally was (1, 1) corresponded to the new point (4, 1). So I multiplied the x-coordinates of the points (1, 1), (2, 2), and (3, 3) on $y = |x|$ by 4 to find the points (4, 1), (8, 2), and (12, 3) on the new graph.



I joined these points to the invariant point (0, 0) to graph one-half of the stretched absolute value function. I used symmetry to complete the graph.

To graph $y = |-x - 3|$, reflect the parent function graph in the y-axis, then translate it 3 units left.



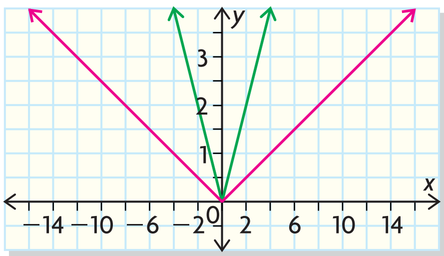
First I thought about the graph of $y = |-x|$. For this graph, I switched the values of x and $-x$, so I really reflected the graph of $y = |x|$ in the y-axis. The points that were originally $(-1, 1)$, $(0, 0)$, and $(1, 1)$ changed to $(1, 1)$, $(0, 0)$, and $(-1, 1)$; so the graph didn't change.

Next, I reasoned that since $|-x| = |x|$, $|-x - 3| = |x + 3|$. Therefore, I shifted the graph of $y = |x|$ 3 units left. The points that were originally $(-1, 1)$, $(0, 0)$, and $(1, 1)$ changed to $(-4, 1)$, $(-3, 0)$, and $(-2, 1)$.

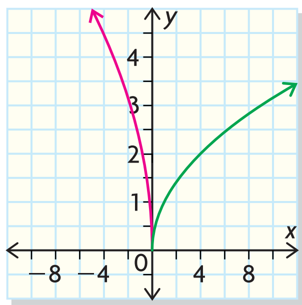
EXAMPLE 2 Using a graph to determine the equation of a transformed function

In the graphs shown, three parent functions have been graphed in green. The functions graphed in red have equations of the form $y = f(kx)$. Determine the equations. Explain your reasoning.

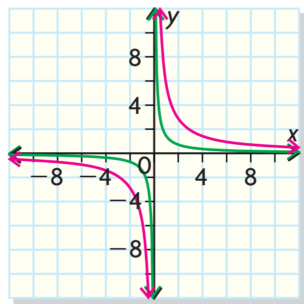
a)



b)



c)



Robert's Solution

a) The parent function is $f(x) = |x|$.

I recognized the V shape of the absolute value function.

Point $(1, 1)$ on $y = |x|$ corresponds to point $(4, 1)$ on the red graph.

I knew that $y = f(kx)$ represents a horizontal stretch or compression and/or a reflection in the y-axis.

Point (2, 2) corresponds to point (8, 2), and point (3, 3) corresponds to point (12, 3).

The red graph is a stretched-out version of the green graph, so k must be between 0 and 1.

The red graph is the green graph stretched horizontally by the factor 4.

The x-coordinates of points on the red graph are 4 times the ones on the green graph.

The equation is $y = |\frac{1}{4}x|$.

Since the stretch scale factor is 4, and $0 < k < 1$, it follows that $k = \frac{1}{4}$. So I could complete the equation.

b) The green graph is a graph of the square root function $f(x) = \sqrt{x}$.

The green graph is the square root function because it begins at (0, 0) and has the shape of a half parabola on its side.

The green graph has been compressed horizontally and reflected in the y-axis to produce the red graph.

The red graph is a compressed version of the green graph that had been flipped over the y-axis. Therefore, k is negative and less than -1.

$(1, 1)$ corresponds to $(-0.25, 1)$.

$(4, 2)$ corresponds to $(-1, 2)$.

$(16, 4)$ corresponds to $(-4, 4)$.

Each x-coordinate has been divided by -4.

The equation is $y = \sqrt{-4x}$.

I divided the corresponding x-coordinates to find k:

$1 \div -0.25 = -4$

$4 \div -1 = -4$

$16 \div -4 = -4$, so $k = -4$

c) The parent function is $f(x) = \frac{1}{x}$.

I recognized the reciprocal function because the graph was in two parts and had asymptotes.

The graph has been stretched horizontally.

The red graph is further away from the asymptotes than the green graph, so it must have been stretched.

$(1, 1)$ corresponds to $(6, 1)$.

$(\frac{1}{2}, 2)$ corresponds to $(3, 2)$.

$(-1, -1)$ corresponds to $(-6, -1)$.

$(-\frac{1}{2}, -2)$ corresponds to $(-3, -2)$.

Each x-coordinate has been multiplied by 6. The equation is $y = \frac{1}{(\frac{1}{6}x)}$.

Since the stretch scale factor is 6, and $0 < k < 1$, it follows that $k = \frac{1}{6}$.

EXAMPLE 3 Using transformations to sketch the graph for a real situation

Use transformations to sketch the graph of the pendulum function $p(L) = 2\pi\sqrt{\frac{1}{10}L}$, where $p(L)$ is the time, in seconds, that it takes for a pendulum to complete one swing and L is the length of the pendulum, in metres.

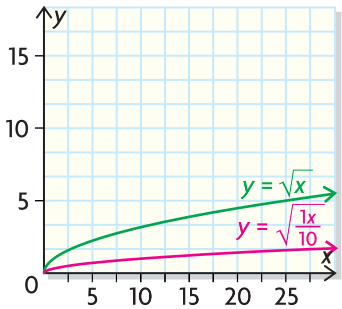
Shannon's Solution

The graph of $y = 2\pi\sqrt{\frac{1}{10}x}$ is the graph of the parent function $y = \sqrt{x}$ stretched horizontally by the factor 10 and vertically by the factor 2π.

The original equation was in the form $y = af(kx)$.

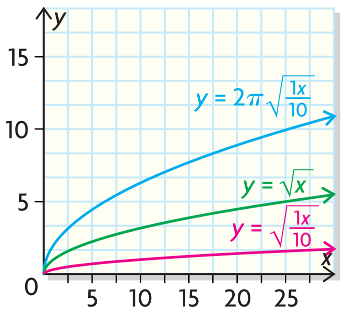
Since $0 < k < 1$, the graph is stretched horizontally by a scale factor of $\frac{1}{\frac{1}{10}} = 10$.

Because $a = 2\pi$ and $2\pi > 1$, the graph is stretched vertically by a scale factor of 2π.



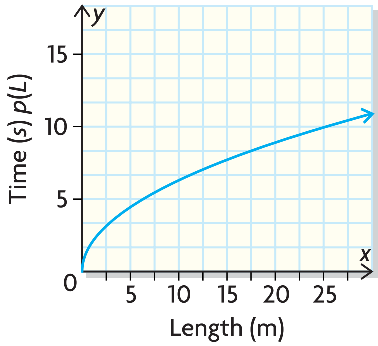
I applied the horizontal stretch.

I multiplied the x-coordinates by 10 to find points on the horizontally stretched graph:
(1, 1) moves to (10, 1).
(4, 2) moves to (40, 2).



Then I applied the vertical stretch to the red graph. I multiplied the y-coordinates by 2π, which is approximately 6.3 (to one decimal place):
(10, 1) moves to (10, 6.3).
(40, 2) moves to (40, 12.6) .

Period versus Length for a Pendulum



I drew a correctly labelled graph of the situation. I copied the sketch onto a graph with length L on the x-axis and time $p(L)$ on the y-axis.

In Summary

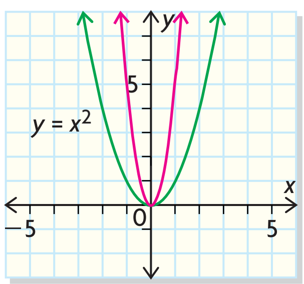
Key Idea

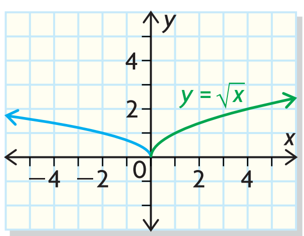
- Functions of the form $g(x) = f(kx)$ have graphs that are not congruent to the graph of $f(x)$. The differences in shape are a result of stretching or compressing in a horizontal direction.

Need to Know

- The image of the point (x, y) on the graph of $f(x)$ is the point $\left(\frac{x}{k}, y\right)$ on the graph of $f(kx)$.
- If $g(x) = f(kx)$, then the value of k has the following effect on the graph of $f(x)$:
  - When $|k| > 1$, the graph is compressed horizontally by the factor $\frac{1}{|k|}$.
  - When $0 < |k| < 1$, the graph is stretched horizontally by the factor $\frac{1}{|k|}$.
  - When $k < 0$, the graph is also reflected in the y-axis.

CHECK *Your Understanding*





1. a) The red graph has been compressed horizontally by the factor $\frac{1}{3}$ relative to the graph of $y = x^2$. Write the equation of the red graph.

b) The blue graph has been stretched horizontally by the factor 2 relative to the graph of $y = \sqrt{x}$ and then reflected in the y-axis. Write the equation of the blue graph.

2. For each function, identify the parent function and describe how the graph of the function can be obtained from the graph of the parent function. Then sketch both graphs on the same set of axes.

a) $y = |0.5x|$ **b)** $y = \left(\frac{1}{4}x\right)^2$ **c)** $y = \sqrt{-2x}$ **d)** $y = \frac{1}{(5x)}$

PRACTISING

3. The point (3, 4) is on the graph of $y = f(x)$. State the coordinates of the image of this point on each graph.

a) $y = f(2x)$ **b)** $y = f(0.5x)$ **c)** $y = f\left(\frac{1}{3}x\right)$ **d)** $y = f(-4x)$

4. Sketch graphs of each pair of transformed functions, along with the graph of the parent function, on the same set of axes. Describe the transformations in words and note any invariant points.

a) $y = (2x)^2, y = (5x)^2$

b) $y = \sqrt{3x}, y = \sqrt{4x}$

c) $y = \frac{1}{(2x)}, y = \frac{1}{(3x)}$

d) $y = |3x|, y = |5x|$

5. Repeat question 4 for each pair of transformed functions.

a) $y = (-2x)^2, y = (-5x)^2$

b) $y = \sqrt{-3x}, y = \sqrt{-4x}$

c) $y = \frac{1}{(-2x)}, y = \frac{1}{(-3x)}$

d) $y = |-3x|, y = |-5x|$

6. Repeat question 4 for each pair of transformed functions.

a) $y = \left(\frac{1}{2}x\right)^2, y = \left(\frac{1}{3}x\right)^2$

b) $y = \sqrt{\frac{1}{2}x}, y = \sqrt{\frac{1}{3}x}$

c) $y = \frac{1}{(\frac{1}{2}x)}, y = \frac{1}{(\frac{1}{4}x)}$

d) $y = \left|\frac{1}{3}x\right|, y = \left|\frac{1}{5}x\right|$

7. Repeat question 4 for each pair of transformed functions.

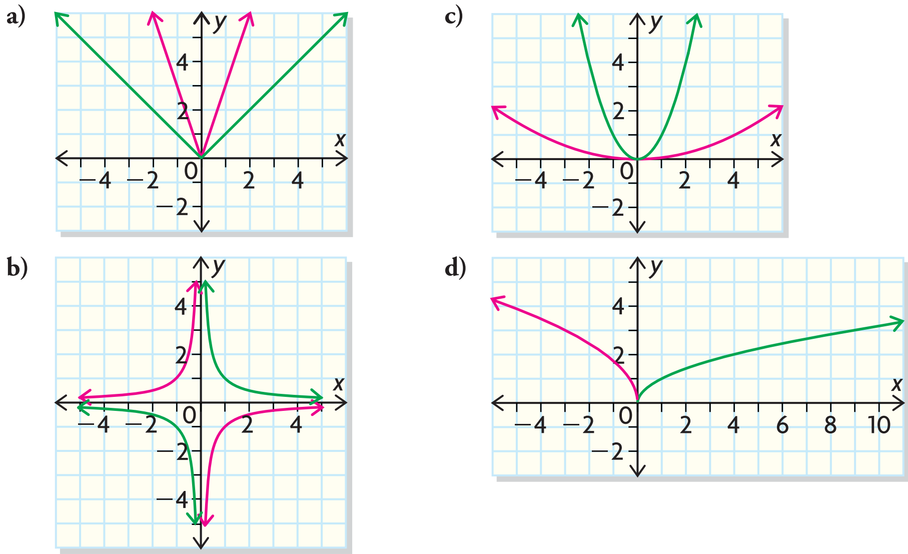
a) $y = \left(-\frac{1}{2}x\right)^2, y = \left(-\frac{1}{3}x\right)^2$

b) $y = \sqrt{-\frac{1}{2}x}, y = \sqrt{-\frac{1}{3}x}$

c) $y = \frac{1}{(-\frac{1}{2}x)}, y = \frac{1}{(-\frac{1}{4}x)}$

d) $y = \left|-\frac{1}{3}x\right|, y = \left|-\frac{1}{5}x\right|$

8. **K** In each graph, one of the parent functions $f(x) = x^2, f(x) = \sqrt{x}, f(x) = \frac{1}{x}$, and $f(x) = |x|$ has undergone a transformation of the form $f(kx)$. Determine the equations of the transformed functions graphed in red.



9. **A** When an object is dropped from a height, the time it takes to reach the ground is a function of the height from which it was dropped. An equation for this function is $t(h) = \sqrt{\frac{h}{4.9}}$, where h is in metres and t is in seconds.
   a) Describe the domain and range of the function.
   b) Sketch the graph by applying a transformation to the graph of $t(h) = \sqrt{h}$.

10. For each set of functions, transform the graph of $f(x)$ to sketch $g(x)$ and $h(x)$, and state the domain and range of each function.
   a) $f(x) = x^2, g(x) = \left(\frac{1}{4}x^2\right), h(x) = (-4x^2)$
   b) $f(x) = \sqrt{x}, g(x) = \sqrt{\frac{1}{5}x}, h(x) = \sqrt{-5x}$
   c) $f(x) = \frac{1}{x}, g(x) = \frac{1}{4x}, h(x) = \frac{1}{(-\frac{1}{3}x)}$
   d) $f(x) = |x|, g(x) = |-2x|, h(x) = \left|\frac{1}{2}x\right|$

11. The function $y = f(x)$ has been transformed to $y = f(kx)$. Determine the value of k for each transformation.
   a) a horizontal stretch by the factor 4
   b) a horizontal compression by the factor $\frac{1}{2}$
   c) a reflection in the y-axis
   d) a horizontal compression by the factor $\frac{1}{5}$ and a reflection in the y-axis

12. **T** A quadratic function has equation $f(x) = x^2 - x - 6$. Determine the x-intercepts for each function.
   a) $y = f(2x)$
   b) $y = f\left(\frac{1}{3}x\right)$
   c) $y = f(-3x)$

13. **C** a) Describe how the graph of $y = f(kx)$ can be obtained from the graph of $y = f(x)$. Include examples that show how the transformations vary with the value of k.
   b) Compare the graph of $y = f(kx)$ with the graph of $y = kf(x)$ for different values of k and different functions $f(x)$. How are the transformations alike? How are they different?

Extending

14. a) Graph the function $y = \frac{1}{x}$.
   b) Apply a horizontal stretch with factor 2.
   c) Apply a vertical stretch with factor 2. What do you notice?
   d) Write the equations of the functions that result from the transformations in parts (b) and (c). Explain why these equations are the same.

15. Suppose you are asked to graph $y = f(2x + 4)$. What two transformations are required? Does the order in which you apply these transformations make a difference? Choose one of the parent functions and investigate. If you get two different results, use a graphing calculator to verify which graph is correct.