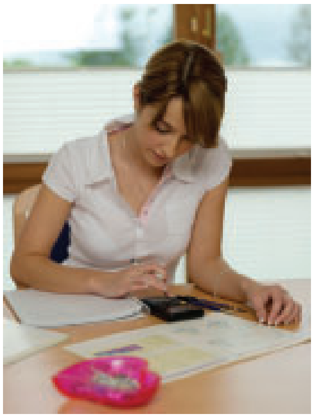
2.2 Factoring Polynomials: Common Factoring

YOU WILL NEED
- graph paper

GOAL

Factor polynomials by dividing out the greatest common factor.



LEARN ABOUT the Math

Elvira squared the numbers 3, 4, and 5 and then added 1 to get a sum of 51.

$$\begin{aligned} 3^2 + 4^2 + 5^2 + 1 &= 9 + 16 + 25 + 1 \\ &= 51 \end{aligned}$$

She repeated this process with the numbers 8, 9, and 10 and got a sum of 246.

$$\begin{aligned} 8^2 + 9^2 + 10^2 + 1 &= 64 + 81 + 100 + 1 \\ &= 246 \end{aligned}$$

Both of her answers are divisible by 3.

$$\frac{51}{3} = 17 \quad \text{and} \quad \frac{246}{3} = 82$$

? Is one more than the sum of the squares of three consecutive integers always divisible by 3?

EXAMPLE 1 Selecting a strategy to determine the greatest common factor

Ariel's Solution: Using Algebra

$$n^2 + (n+1)^2 + (n+2)^2 + 1$$

I let the three consecutive integers be n, $n + 1$, and $n + 2$, where n represents the first integer. I then wrote an expression for the sum of their squares and added 1.

$$\begin{aligned} &= n^2 + (n+1)(n+1) + (n+2)(n+2) + 1 \\ &= n^2 + (n^2 + 2n + 1) + (n^2 + 4n + 4) + 1 \\ &= 3n^2 + 6n + 6 \end{aligned}$$

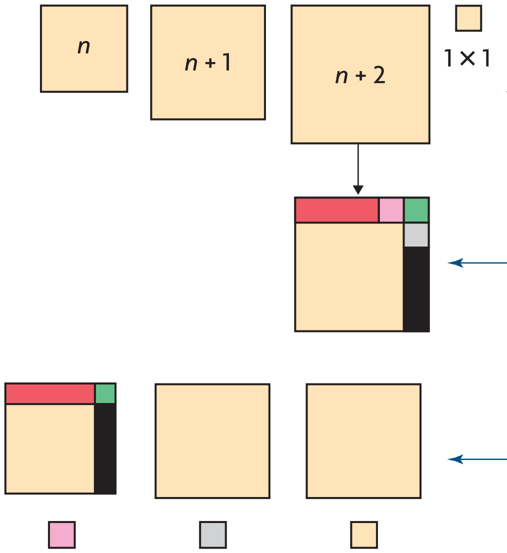
I simplified the resulting expression by squaring and collecting like terms.

$$= 3(n^2 + 2n + 2)$$

I factored the polynomial by finding the greatest common factor (GCF) of its terms: 3. I divided it out.

Three is a factor of the expression. This shows that one more than the sum of the squares of three consecutive integers is always divisible by 3.

David's Solution: Using Area Models



From construction paper, I made three squares, each 1 unit longer than the next.

These represent the squares of any three consecutive integers. I also made a 1×1 square.

I made straight cuts in the largest square to create five pieces: three equal squares and two rectangles.

I moved three of those pieces to the smallest square to make it as big as the middle square.

Altogether, I had three groups, each with one large square and one small square. The number represented by each group is $(n + 1)^2 + 1$.

Since there are three equal groups, the total value is a multiple of 3. This shows that one more than the sum of the squares of three consecutive integers is always divisible by 3.

Reflecting

A. Why did Ariel introduce a variable to solve this problem?

B. Why did Ariel want to write the expression with a factor of 3?

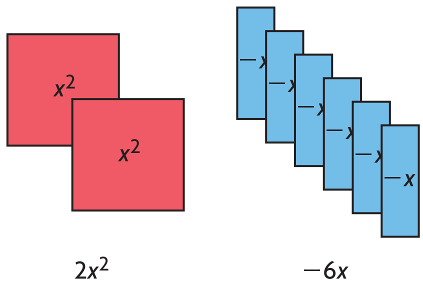
C. Ariel used the expression $3(n^2 + 2n + 2)$ to show that the sum was divisible by 3. David used $3[(n + 1)^2 + 1]$. Are the two expressions equivalent? Explain.

D. What are the advantages and disadvantages of each method of showing divisibility?

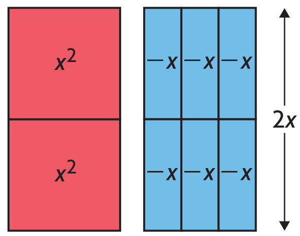
EXAMPLE 2 Selecting a strategy to represent the greatest common factor

Factor $2x^2 - 6x$.

Chloe's Solution: Using Algebra Tiles



I used algebra tiles to represent each term, red tiles for positives and blue tiles for negatives.



To get the greatest common factor of $2x^2$ and $-6x$, I rearranged the two terms separately into two rectangles with width $2x$ for each term. This is the greatest width I could use with the x^2 tiles.

The greatest common factor of the terms of $2x^2 - 6x$ is $2x$.

$2x^2 - 6x = 2x(x - 3)$

To find the other factor, I looked at my model to determine the length. It was $x - 3$.

Luis's Solution: Using Symbols

$2x^2 - 6x = 2(x^2) + 2(-3)x$

The greatest common factor of the coefficients 2 and 6 is 2.

$= 2x(x) + 2x(-3)$

The greatest common factor of the variable parts x^2 and x is x.

The greatest common factor of the terms of $2x^2 - 6x$ is $2x$.

$2x^2 - 6x = 2x(x - 3)$

I used the distributive property to write the product of the factors.

Sometimes algebraic expressions have a binomial as a common factor.

EXAMPLE 3 Using reasoning to factor

Factor $3x(x + 1) - 2(x + 1)$.

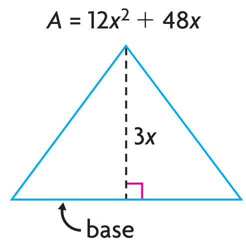
Ida's Solution

$$3x(x + 1) - 2(x + 1) = (3x - 2)(x + 1)$$

Both $3x$ and -2 are being multiplied by $x + 1$, so $x + 1$ is a common factor. I wrote an equivalent expression using the distributive property.

EXAMPLE 4 Solving a problem by factoring

A triangle has an area of $12x^2 + 48x$ and a height of $3x$. What is the length of its base?



Aaron's Solution

$$A = \frac{1}{2}bh$$

The formula for the area of a triangle is $A = \frac{1}{2}bh$.

$$12x^2 + 48x = \frac{1}{2}b \times 3x$$

I knew A and h; I had to solve for b.

$$3x(4x + 16) = \frac{1}{2}b \times 3x$$

I factored $3x$ out of the terms on the left side.

$$4x + 16 = \frac{1}{2}b$$

I divided both sides of the equation by $3x$. If $x = 0$, I can't carry out this operation, but if $x = 0$, then the triangle doesn't exist.

$$2(4x + 16) = \left(\frac{1}{\cancel{2}_1}b\right)\cancel{2}^1$$

I multiplied both sides of the equation by 2.

$$8x + 32 = b$$

The length of the base is $8x + 32$, where $x \neq 0$.

In Summary

Key Ideas

- Factoring algebraic expressions is the opposite of expanding. Expanding involves multiplying, while factoring involves looking for the expressions to multiply. For example:

$$\xrightarrow{\text{expanding}}$$
$$2x(3x - 5) = 6x^2 - 10x$$
$$\xleftarrow[\text{factoring}]{}$$

- One way to factor a polynomial is to look for the greatest common factor of its terms as one of its factors. For example, $6x^2 + 2x - 4$ can be factored as $2(3x^2 + x - 2)$, since 2 is the greatest common factor of each term.

Need to Know

- It is possible to factor a polynomial by dividing by a common factor that is not the greatest common factor. This will result in another polynomial that still has a common factor. For example:

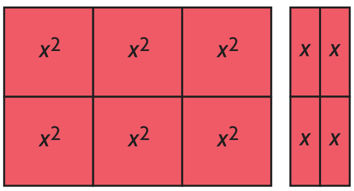
$$\begin{aligned} 4x + 8 &= 2(2x + 4) \\ &= 2(2)(x + 2) \\ &= 4(x + 2) \end{aligned}$$

- A polynomial is factored fully when only 1 or -1 is a common factor of every term.
- A common factor can have any number of terms. For example, a common factor of $8x^2 + 6x$ is $2x$, a monomial. But a common factor of $(2x - 1)^2 - 4(2x - 1)$ is $(2x - 1)$, a binomial.

CHECK Your Understanding

1. In each diagram, two terms of a polynomial have been rearranged to show their common factor. For each, identify the terms of the polynomial and the common factor.

a)



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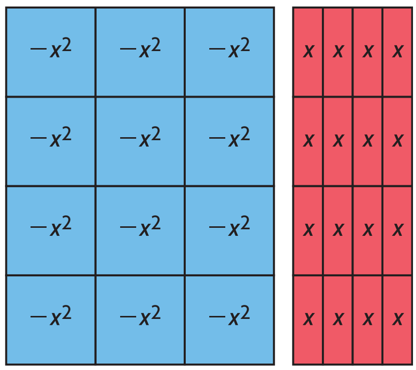


2. Name the common factor of the terms of the polynomial.

a) $3x^2 - 9x + 12$ **b)** $5x^2 + 3x$

3. Factor, using the greatest common factor.

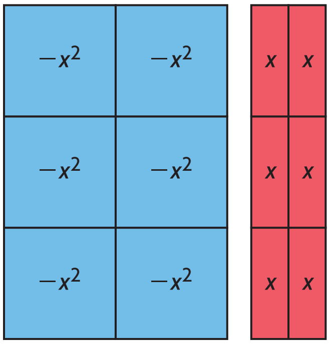
a) $4x^2 - 6x + 2$ **c)** $5a(a + 7) + 2(a + 7)$

b) $5x^2 - 20x$ **d)** $4m(3m - 2) - (3m - 2)$

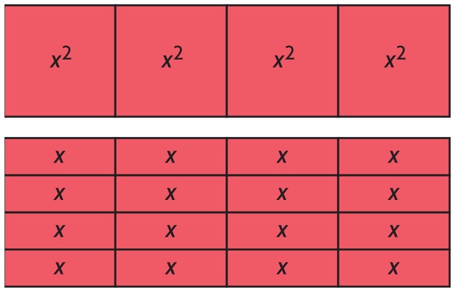
PRACTISING

4. In each diagram, two terms of a polynomial have been rearranged to show their common factor. For each, identify the terms of the polynomial and the common factor.

a)



b)



5. Name the greatest common factor of each polynomial.

a) $6x^2 + 12x - 18$ **c)** $16x^2 - 8x + 10$

b) $4x^2 + 14x$ **d)** $-15x^2 - 10$

6. Factor.

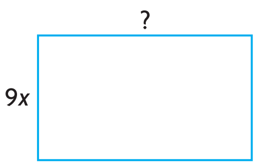
K **a)** $27x^2 - 9x$ **d)** $-2a^2 - 4a + 6$

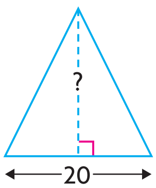
b) $-8m^2 + 20m$ **e)** $3x(x + 7) - 2(x + 7)$

c) $10x^2 - 5x + 25$ **f)** $x(3x - 2) + (3x - 2)(x + 1)$

7. The area, A, of each figure is given. Determine the unknown measurement.

a) $A = 18x^2 - 9x$ **b)** $A = 10m^2 - 20m + 20$





8. The formula for the surface area of a cylinder is $SA = 2\pi r^2 + 2\pi rh$.
A A cylinder has a height of 10 units and a radius of r units. Determine a factored expression for its total surface area.

9. Show that $(3a - 2)$ is a common factor of $2a(3a - 2) + 7(2 - 3a)$.

10. **a)** Give three examples of a quadratic binomial with greatest common factor $6x$, and then factor each one.

b) Give three examples of a quadratic trinomial with greatest common factor 7, and then factor each one.

11. Colin says that the greatest common factor of $-8x^2 + 4x - 6$ is 2, but Colleen says that it is -2. Explain why both answers could be considered correct.

12. **T** For what values of k is it possible to divide out a common factor from $6x^2 + kx - 12$, but not from $6x^2 + kx + 4$? Explain.

13. It seems that the sum of the squares of two consecutive even or odd integers is always even. For example:

$$\begin{aligned} 4^2 + 6^2 &= 16 + 36 \\ &= 52 \\ 7^2 + 9^2 &= 49 + 81 \\ &= 130 \end{aligned}$$

Let n represent the first integer.

a) What expression represents the second integer?

b) What expression represents the sum of the two squares?

c) Use algebra to show that the result is always even.

14. **C** How does knowing how to determine the greatest common factor of two numbers help you factor polynomials?

Extending

15. Factor.

a) $5x^2y - 10xy^2$

b) $10a^2b^3 + 20a^2b - 15a^2b^2$

c) $3x(x + y) - y(x + y)$

d) $5y(x - 2) - 7(2 - x)$

16. A factor might be common to only some terms of a polynomial, but grouping these terms sometimes allows the polynomial to be factored. For example:

$$\begin{aligned} &ax - ay + bx - by \\ &= a(x - y) + b(x - y) \\ &= (x - y)(a + b) \end{aligned}$$

Now you see the common factor $(x - y)$.

Factor these expressions by grouping.

a) $9xa + 3xb + 6a + 2b$

b) $10x^2 - 5x - 6xy + 3y$

c) $(x + y)^2 + x + y$

d) $1 + xy + x + y$