1.6 Using Multiple Transformations to Graph Quadratic Functions

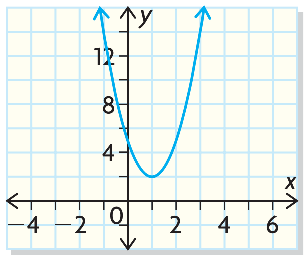
GOAL

Apply multiple transformations to $f(x) = x^2$ to graph quadratic functions defined by $f(x) = a(x - h)^2 + k$.

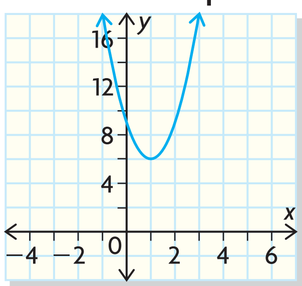
LEARN ABOUT the Math

Dave and Nathan have each graphed the function $f(x) = 3(x - 1)^2 + 2$. Dave's graph is different from Nathan's graph. They both applied the transformations in different orders.

Dave's Graph



Nathan's Graph



? Does the order in which transformations are performed matter?

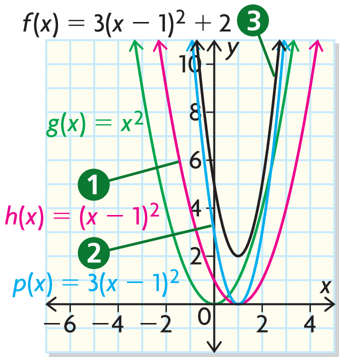
EXAMPLE 1 Reasoning about order: following the order of operations

Dave's Solution: Stretching and then Translating

$f(x) = 3(x - 1)^2 + 2$

$a = 3, h = 1, k = 2$

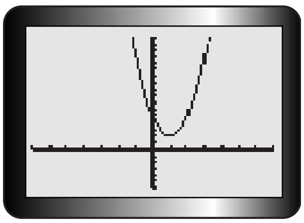
I began with $g(x) = x^2$. I applied the transformations by following the order of operations.



1 I performed the operation inside the bracket first. I translated the graph 1 unit to the right to get the graph $h(x) = (x - 1)^2$ (in red).

2 Next, I performed the multiplication. The translated graph stretches vertically by a factor of 3 to get the graph $p(x) = 3(x - 1)^2$ (in blue).

3 I translated the stretched graph 2 units up to get the graph $f(x) = 3(x - 1)^2 + 2$ (in black).



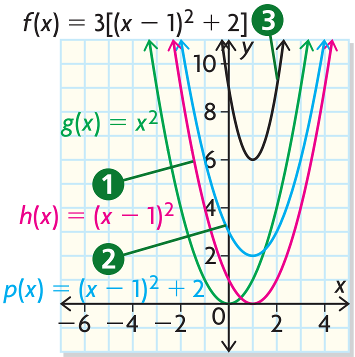
I checked my graph on the graphing calculator. My graph looks the same. The order I used must be correct.

Nathan's Solution: Translating and then Stretching

$f(x) = 3(x - 1)^2 + 2$

$a = 3, h = 1, k = 2$

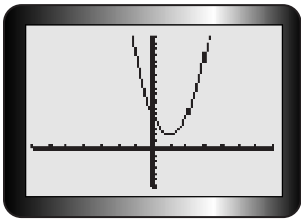
I began with $g(x) = x^2$. I applied the translations first.



1 I shifted the graph 1 unit to the right, since $h = 1$, to get the graph $h(x) = (x - 1)^2$ (in red).

2 I translated the graph 2 units up, since $k = 2$, to get the graph $p(x) = (x - 1)^2 + 2$ (in blue).

3 Next, I applied the vertical stretch. The graph stretches vertically by a factor of 3, since $a = 3$, to get the graph $f(x) = 3[(x - 1)^2 + 2]$ (in black).



I checked my graph on the graphing calculator. My graph looks different. The order I used can't be correct.

Reflecting

A. List the y-coordinates for $x = -2, -1, 0, 1$, and 2 for the final graph of each student's solution.

B. List the y-coordinates for $x = -2, -1, 0, 1$, and 2 for the graph of the function $f(x) = 3(x - 1)^2 + 2$.

C. How do the coordinates from part B compare with those from each student's solution?

D. Explain why one of the solutions is incorrect.

E. List the order of transformations of the correct solution. Does the order of transformations of the correct solution apply to all functions of the form $f(x) = a(x - h)^2 + k$?

APPLY the Math

EXAMPLE 2 | Applying multiple transformations to sketch the graph of a quadratic function

Sketch $h(x) = 2(x + 3)^2 - 1$ by applying the appropriate transformations to the graph of $f(x) = x^2$.

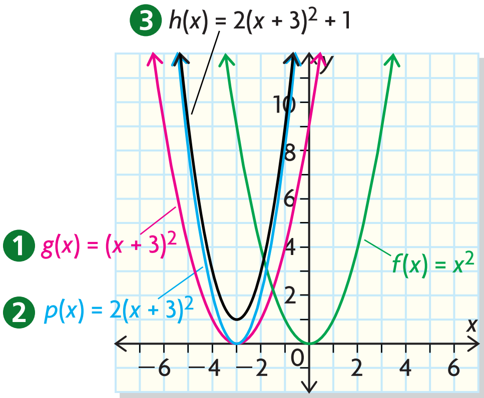
Mei's Solution

$h(x) = 2(x + 3)^2 + 1$

$h(x) = 2(x - (-3))^2 + 1$

$a = 2, h = -3, k = 1$

I applied the transformations one at a time, following the order of operations.



1. I moved the graph of $f(x) = x^2$ (in green) 3 units to the left to get the graph of $g(x) = (x + 3)^2$ (in red).
2. I multiplied the y-coordinates of $g(x) = (x + 3)^2$ by 2 to get the graph of $p(x) = 2(x + 3)^2$ (in blue).
3. I moved the resulting graph 1 unit up to get the graph of $h(x) = 2(x + 3)^2 + 1$ (in black).

The vertex changed from (0, 0) to (−3, 1).
The axis of symmetry changed from $x = 0$ to $x = -3$.
The final graph is narrower than $f(x) = x^2$.

EXAMPLE 3 | Identifying transformations from the quadratic function

Describe the transformations you would use to graph the function $f(x) = -(x - 2.5)^2 - 5$.

Deirdre's Solution

$f(x) = -(x - 2.5)^2 - 5$

$a = -1, h = 2.5, k = -5$

Step 1.	Horizontal translation 2.5 units to the right	$h = 2.5$
Step 2.	No vertical stretch or compression	$a = -1$
Step 3.	Reflection in the x-axis	$a < 0$
Step 4.	Vertical translation 5 units down	$k = -5$

EXAMPLE 4 Applying multiple transformations to sketch the graph of a quadratic function

Graph $g(x) = -7 - (x + 3)^2$ by using transformations.

Jared's Solution

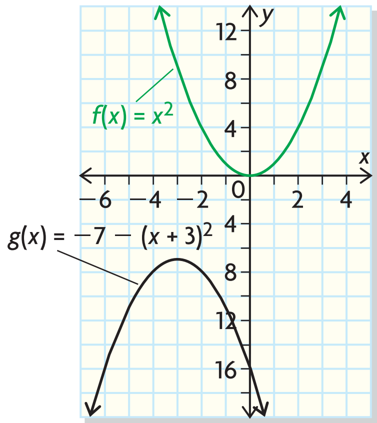
$g(x) = -7 - (x + 3)^2$

$g(x) = -(x + 3)^2 - 7$

$g(x) = -(x - (-3))^2 - 7$

$a = -1, h = -3, k = -7$

I wrote the function in the form $g(x) = a(x - h)^2 + k$ to help me identify the values of a, h, and k.



Since $h < 0$, I moved the graph of $f(x) = x^2$ (in green) 3 units to the left. Next, I reflected it in the x-axis because $a = -1$. Then I moved it 7 units down because $k < 0$. This gave me the graph of $g(x)$ (in black).

The vertex changed from (0, 0) to $(-3, -7)$.
The axis of symmetry changed from $x = 0$ to $x = -3$.
The shape did not change, since no stretching or compressing occurred.

In Summary

Key Idea

- Functions of the form $g(x) = a(x - h)^2 + k$ can be graphed by hand by applying the appropriate transformations, one at a time, to the graph of $f(x) = x^2$.

Need to Know

- Transformations can be applied following the order of operations:
  - horizontal translations
  - vertical stretches or compressions
  - reflections, if necessary
  - vertical translations
- You can use fewer steps if you combine the stretch/compression with the reflection and follow this with the necessary translations. This works because you are multiplying before adding/subtracting like the order of operations

CHECK Your Understanding

1. List the sequence of steps required to graph each function.

a) $f(x) = 3(x + 2)^2$

b) $f(x) = -2(x - 3)^2 + 1$

c) $f(x) = \frac{1}{3}x^2 - 3$

d) $f(x) = -\frac{1}{2}(x + 2)^2 + 4$

2. Sketch the final graph for each of the functions in question 1. Verify at least one of the key points, other than the vertex, by substituting its x-value into the equation and solving for y.

PRACTISING

3. Match each function to its graph.

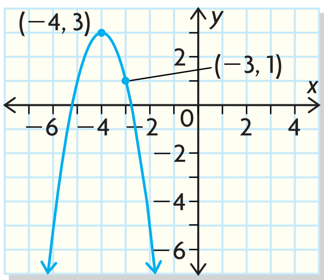
a) $f(x) = (x + 3)^2 + 1$

b) $f(x) = -2(x + 4)^2 + 3$

c) $f(x) = -(x - 3)^2 - 2$

d) $f(x) = \frac{1}{2}(x - 3)^2 - 5$

i)



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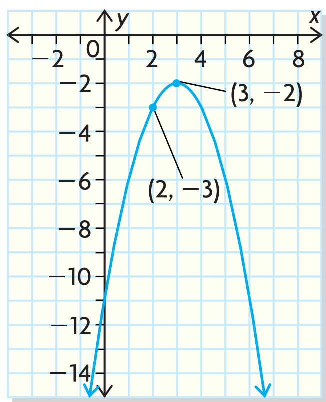

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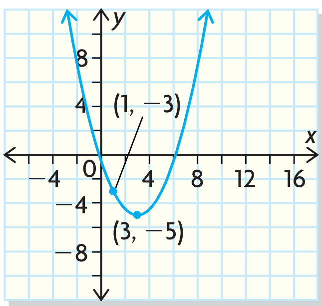

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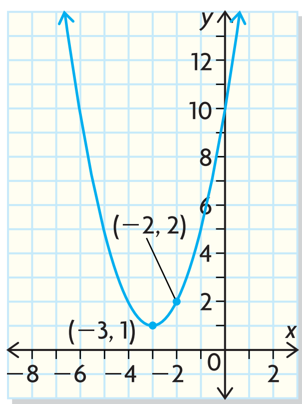
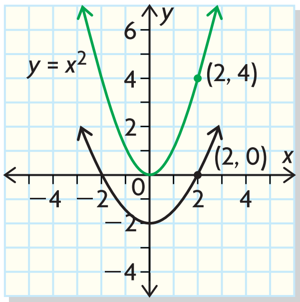
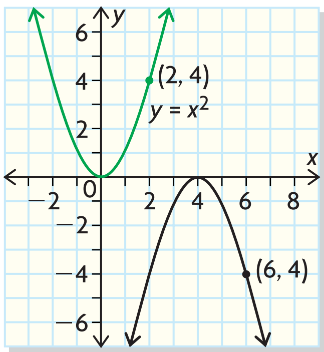


4. Transformations are applied to the graphs of $y = x^2$ to obtain the black parabolas. Describe the transformations that were applied. Write an equation for each black parabola.

a)



b)



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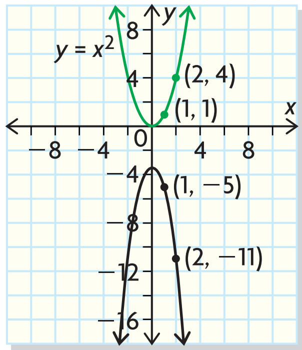
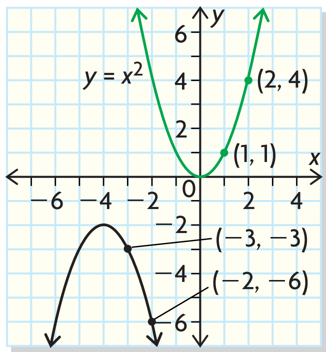


d)



5. Write an equation of a parabola that satisfies each set of conditions.

a) opens upward, congruent with $y = 2x^2$, vertex $(4, -1)$

b) opens downward, congruent with $y = \frac{1}{3}x^2$, vertex $(-2, 3)$

c) opens downward, congruent with $y = \frac{1}{2}x^2$, vertex $(-3, 2)$

d) vertex $(2, -2)$, x-intercepts of 0 and 4

e) vertex $(-1, 4)$, y-intercept of 2

f) vertex $(4, 5)$, passes through $(2, 9)$

6. Consider a parabola P that is congruent to $y = x^2$, opens upward, and has vertex $(2, -4)$. Now find the equation of a new parabola that results if P is

a) stretched vertically by a factor of 5

b) compressed by a factor of $\frac{1}{2}$

c) translated 2 units to the left

d) translated 3 units up

e) reflected in the x-axis and translated 2 units to the right and 4 units down

7. Describe the transformations applied to the graph of $y = x^2$ to obtain a graph of each quadratic relation. Sketch the graph by hand. Start with the graph of $y = x^2$ and use the appropriate transformations.

a) $y = -4(x - 5)^2 + 3$

b) $y = 2(x + 1)^2 - 8$

c) $y = \frac{2}{3}(x + 2)^2 + 1$

d) $y = -\frac{1}{2}(x - 1)^2 - 5$

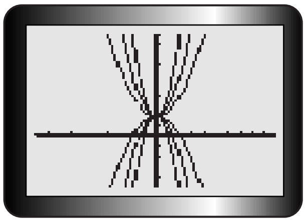
e) $y = -(x - 3)^2 + 2$

f) $y = 2(x + 1)^2 + 4$

8. K Sketch the graph of the transformed function $g(x) = -3(x - 2)^2 + 5$. Start with the graph of $f(x) = x^2$ and use the appropriate transformations.

9. T If $y = x^2$ is the base curve for the graphs shown, what equations could be used to produce these screens on a graphing calculator? The scale on both axes is 1 unit per tick mark.

a)



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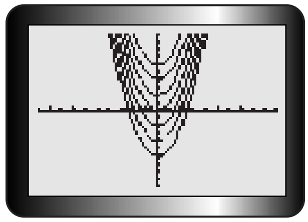
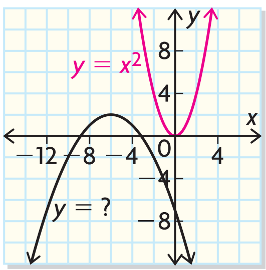




10. A The graphs of $y = x^2$ (in red) and another parabola (in black) are shown at the left.

a) Determine a combination of transformations that would produce the second parabola from the first.

b) Determine a possible equation for the second parabola.

11. C Describe the transformations applied to the graph of $y = x^2$ to obtain the graph of each quadratic relation.

a) $y = 2(x + 7)^2 - 3$

b) $y = 2x^2 + 7$

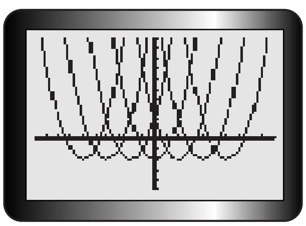
c) $y = -3(x - 4)^2 + 2$

d) $y = -3x^2 - 4$

Extending

12. A graphing calculator was used together with the vertex form $y = a(x - h)^2 + k$ to graph the screens shown. For the set of graphs on each screen, tell which of the variables a, h, and k remained constant and which changed. Give possible values for the variables that remained constant.

a)



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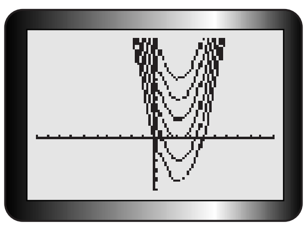

c)