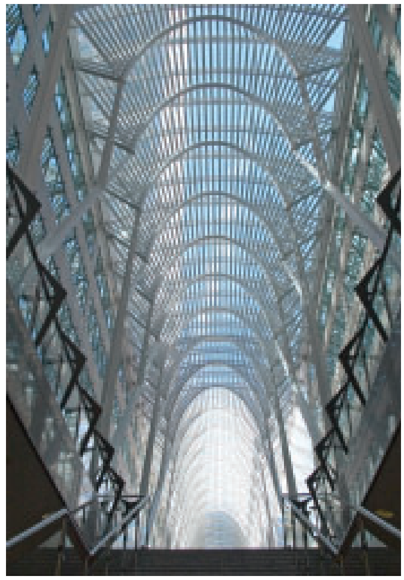
1.5 Graphing Quadratic Functions by Using Transformations

GOAL

Use transformations to sketch the graphs of quadratic functions.

LEARN ABOUT the Math



This photograph shows the interior of BCE Place in Toronto. Architects design structures that involve the quadratic model because it combines strength with elegance.

You have seen how changing the values of a, h, and k changes the shape and position of the graph of $f(x)$ in functions of the form $f(x) = x^2 + k$, $f(x) = (x - h)^2$, and $f(x) = ax^2$. This information can be used together with the properties of the quadratic function $f(x) = x^2$ to sketch the graph of the **transformed function**.

transformed function

the resulting function when the shape and/or position of the original graph of $f(x)$ are changed

? How do you use transformations and the properties of the quadratic function $f(x) = x^2$ to graph the function $g(x) = (x + 2)^2 - 4$?

EXAMPLE 1 Graphing quadratic functions by using a transformation strategy: translating

Use transformations to sketch the graph of $g(x) = (x + 2)^2 - 4$.

Dave's Solution

$g(x) = (x + 2)^2 - 4$

$g(x) = (x - (-2))^2 - 4$

$a = 1, h = -2, k = -4$

I expressed the relation in the form $g(x) = a(x - h)^2 + k$.

The values of h and k show that I must apply a horizontal and a vertical **translation** to the graph of $f(x) = x^2$ by moving each point on this graph 2 units to the left and 4 units down.

translation

two types of translations can be applied to the graph of a function:

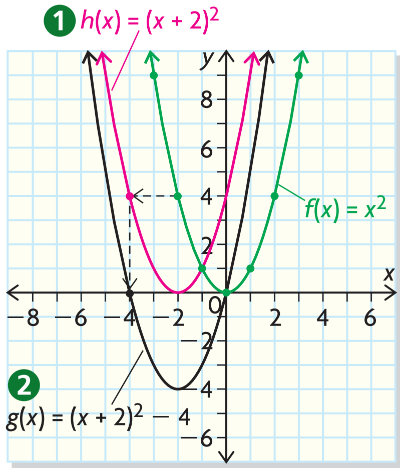
- Horizontal translations—all points on the graph move to the right when $h > 0$ and to the left when $h < 0$
- Vertical translations—all points on the graph move up when $k > 0$ and down when $k < 0$

key points

points of any function that define its general shape

Key Points of $f(x) = x^2$

x	$f(x) = x^2$
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9



1 I subtracted 2 from the x-coordinates of the **key points**. The base curve $f(x) = x^2$ (in green) moved 2 units to the left. The resulting graph is $h(x) = (x + 2)^2$ (in red).

2 I subtracted 4 from the y-coordinates of the key points of $h(x) = (x + 2)^2$. Each point of the curve $h(x) = (x + 2)^2$ moves down 4 units. The resulting graph is $g(x) = (x + 2)^2 - 4$ (in black).

The vertex changed from $(0, 0)$ to $(-2, -4)$.

The axis of symmetry changed from $x = 0$ to $x = -2$.

The shape of the graph did not change.

Reflecting

A. Consider Dave's solution. Given three points $O(0, 0)$, $A(-2, 4)$, and $B(1, 1)$ on the graph of $f(x) = x^2$, what would the coordinates of the corresponding images of the points on $f(x) = (x + 2)^2 - 4$ be if they were labelled O', A', and B'?

B. Dave translated the graph of $f(x) = x^2$ to the left 2 units and then 4 units down. Had he translated the graph 4 units down and then 2 units to the left, would the resulting graph be the same? Explain.

C. If the graph of $f(x) = x^2$ is only translated up/down and left/right, will the resulting graph always be congruent to the original graph? Explain.

APPLY the Math

EXAMPLE 2 Graphing quadratic functions by using a transformation strategy: stretching vertically

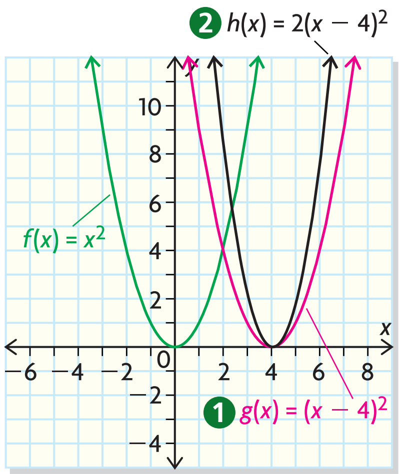
Use transformations to sketch the graph of $h(x) = 2(x - 4)^2$.

Amanda's Solution

$h(x) = 2(x - 4)^2$

$a = 2$, $h = 4$, and $k = 0$.

$h(x)$ is in the form $h(x) = a(x - h)^2 + k$. No vertical translation is required, because $k = 0$.



1. Since $h > 0$, I added 4 to the x-coordinates of the key points of the graph $f(x) = x^2$ (in green). This moved the graph 4 units to the right. The resulting graph is $g(x) = (x - 4)^2$ (in red).
2. I multiplied the y-coordinates of the key points of $g(x) = (x - 4)^2$ by 2 because $a = 2$. This resulted in a **vertical stretch** and gave me the graph of $h(x) = 2(x - 4)^2$ (in black).

The vertex changed from (0, 0) to (4, 0).

The axis of symmetry changed from $x = 0$ to $x = 4$.

The shape of the parabola also changed—it's narrower than $f(x) = x^2$.

vertical stretch

when $a > 1$, the graph of the function $f(x)$ is stretched vertically

EXAMPLE 3 Graphing quadratic functions by using transformation strategies: reflecting and compressing vertically

Use transformations to sketch the graph of $g(x) = -0.5(x + 2)^2$.

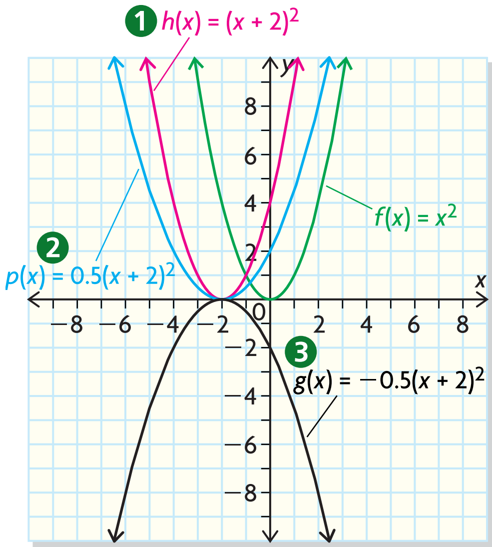
Chantelle's Solution

$g(x) = -0.5(x + 2)^2$

$g(x) = -0.5(x - (-2))^2 + 0$

$a = -0.5$, $h = -2$, and $k = 0$.

I wrote $g(x)$ in the form $g(x) = a(x - h)^2 + k$. No vertical translation is required because $k = 0$.



1. I subtracted 2 from each of the x-coordinates of the key points. The graph of $f(x) = x^2$ (in green) moved 2 units to the left, since $h < 0$. This resulted in the graph of $h(x) = (x + 2)^2$ (in red).

2. I multiplied the y-coordinates of the key points of $h(x) = (x + 2)^2$ by 0.5 to get the graph of $p(x) = 0.5(x + 2)^2$. This resulted in a **vertical compression** and gave me the graph of $p(x) = 0.5(x + 2)^2$ (in blue).

3. Since a is also negative, I reflected the graph of $p(x) = 0.5(x + 2)^2$ in the x-axis to get the graph $g(x) = -0.5(x + 2)^2$ (in black). This resulted in a **vertical reflection**.

The vertex changed from $(0, 0)$ to $(-2, 0)$.

The axis of symmetry changed from $x = 0$ to $x = -2$.

The shape of the parabola also changed—it's wider than $f(x) = x^2$.

The parabola opens downward.

vertical compression

when $0 < a < 1$, the graph is compressed vertically

vertical reflection

when $a < 0$, the graph is reflected in the x-axis

EXAMPLE 4 Graphing quadratic functions by using transformation strategies

Use transformations to sketch the graph of $m(x) = \frac{1}{3}x^2 + 2$.

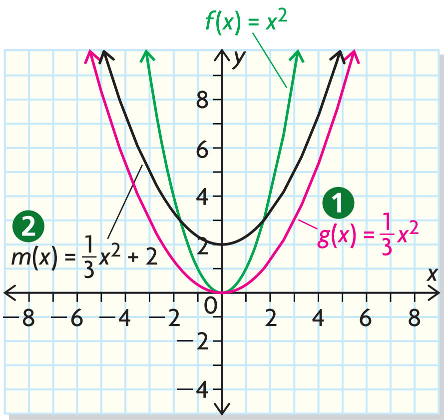
Craig's Solution

$$m(x) = \frac{1}{3}x^2 + 2$$

$$m(x) = \frac{1}{3}(x - 0)^2 + 2$$

I wrote $m(x)$ in the form $m(x) = a(x - h)^2 + k$. No horizontal translation is required because $h = 0$.

$a = \frac{1}{3}$, $h = 0$, and $k = 2$.



1. I multiplied the y-coordinates of $f(x) = x^2$ (in green) by $\frac{1}{3}$. The graph was vertically compressed by a factor of 3 and resulted in the graph of $g(x) = \frac{1}{3}x^2$ (in red).

2. Then I moved each point on $g(x)$ up 2 units. This gave me the graph of $m(x) = \frac{1}{3}x^2 + 2$ (in black).

The vertex changed from (0, 0) to (0, 2).

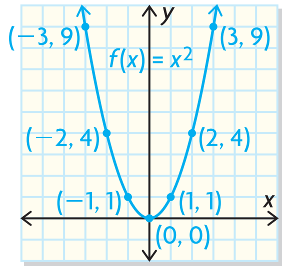
The axis of symmetry remained at $x = 0$.

The shape of the graph also changed—it's wider than $f(x) = x^2$.

In Summary

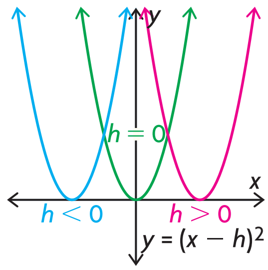
Key Ideas

- Functions of the form $g(x) = a(x - h)^2 + k$ can be graphed by applying transformations, one at a time, to the key points on the graph of $f(x) = x^2$.
- In graphing $g(x)$, the transformations apply to every point on the graph of $f(x)$. However, to sketch the new graph, you only need to apply the transformations to the key points of $f(x) = x^2$.

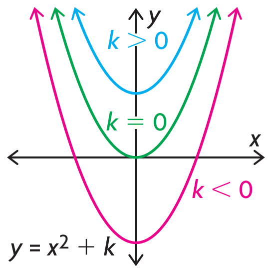


Need to Know

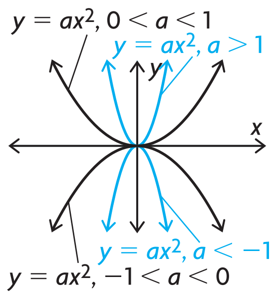
- For a quadratic function in the form $g(x) = a(x - h)^2 + k$,
  - horizontal translations: The graph moves to the right when $h > 0$ and to the left when $h < 0$.



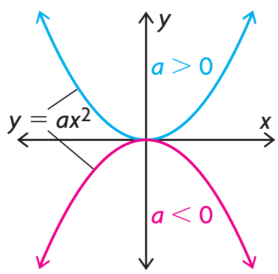
  - vertical translations: The graph moves up when $k > 0$ and down when $k < 0$.



  - vertical stretches: The graph is stretched vertically when $a > 1$.
  - vertical compressions: The graph is compressed vertically when $0 < a < 1$.



  - vertical reflections: The graph is reflected in the x-axis when $a < 0$.
  - the axis of symmetry is the line $x = h$.
  - the vertex is the point (h, k).



CHECK *Your Understanding*

1. Match each equation with its corresponding graph. Explain how you made your decision.

a) $y = -(x - 2)^2 - 3$

b) $y = -0.5x^2 - 4$

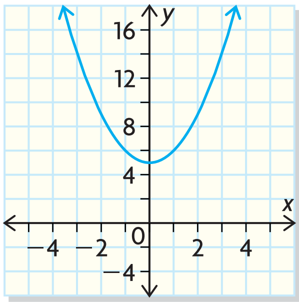
c) $y = x^2 + 5$

d) $y = 2(x + 2)^2$

e) $y = (x - 2)^2$

f) $y = -\frac{1}{3}(x + 4)^2 + 2$

i)



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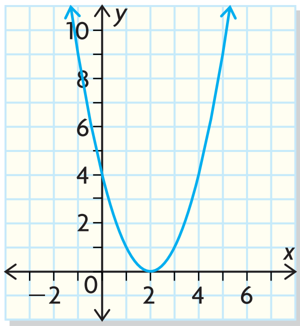

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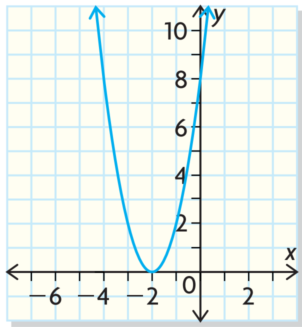

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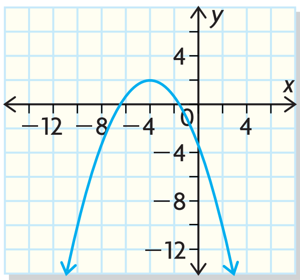

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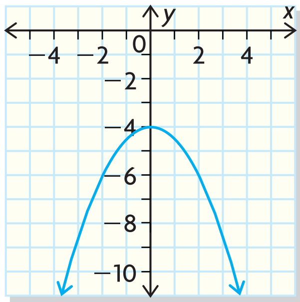

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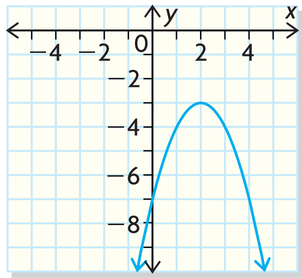


2. For each function,

i) identify the values of the parameters *a*, *h*, and *k*

ii) identify the transformations

iii) use transformations to graph the function and check that it is correct with a table of values or a graphing calculator

a) $f(x) = -3x^2$

b) $f(x) = (x + 3)^2 - 2$

c) $f(x) = (x - 1)^2 + 1$

d) $f(x) = -x^2 - 2$

e) $f(x) = -(x - 2)^2$

f) $f(x) = \frac{1}{2}(x + 3)^2$

PRACTISING

3. Match each graph with the correct equation. The graph of $y = x^2$ is shown in green in each diagram.

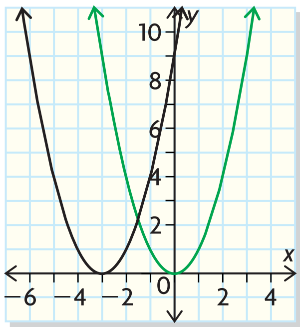
a) $y = x^2 + 5$

b) $y = (x + 5)^2$

c) $y = -2x^2 + 5$

d) $y = 2(x + 5)^2$

i)



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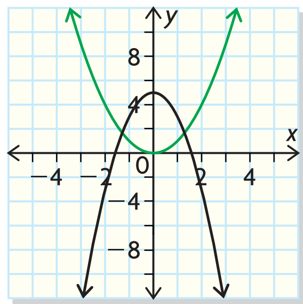

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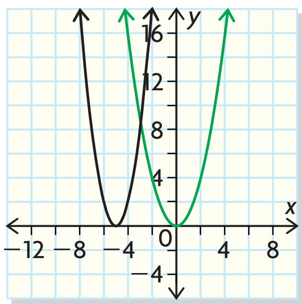

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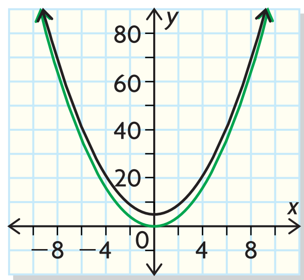


4. **K** Describe the translations applied to the graph of $y = x^2$ to obtain a graph of each quadratic function. Sketch the graph.

a) $y = (x - 5)^2 + 3$

b) $y = (x + 1)^2 - 2$

c) $y = 3x^2 - 4$

d) $y = -\frac{1}{3}(x + 4)^2$

5. Consider a parabola P that is congruent to $y = x^2$ and with vertex at $(0, 0)$. Find the equation of a new parabola that results if P is

a) stretched vertically by a factor of 5

b) compressed vertically by a factor of 2

c) translated 2 units to the right and reflected in the x-axis

d) compressed vertically by 3, reflected in the x-axis, and translated 2 units up

6. Consider a parabola P that is congruent to $y = x^2$ and with vertex $(2, -4)$. Find the equation of a new parabola that results if P is

a) translated 2 units down

b) translated 4 units to the left

c) translated 2 units to the left and translated 3 units up

d) translated 3 units to the right and translate 1 unit down

7. Write an equation of a parabola that satisfies each set of conditions.

a) opens upward, congruent to $y = x^2$, and vertex $(0, 4)$
b) opens upward, congruent to $y = x^2$, and vertex $(5, 0)$
c) opens downward, congruent to $y = x^2$, and vertex $(5, 0)$
d) opens upward, narrower than $y = x^2$, and vertex $(2, 0)$
e) opens downward, wider than $y = x^2$, and vertex $(-2, 0)$
f) opens upward, wider than $y = x^2$, and vertex $(1, 0)$

8. Determine the answers to the following questions for each of the given transformed quadratic functions.

i) How does the shape of the graph compare with the graph of $f(x) = x^2$?
ii) What are the coordinates of the vertex and the equation of the axis of symmetry?
iii) Graph the transformed function and $f(x) = x^2$ on the same set of axes.
iv) Label the points $O(0, 0)$, $A(-2, 4)$, and $B(1, 1)$ on the graph of $f(x) = x^2$. Determine the images of these points on the transformed function. Label the images O', A', and B'.

a) $f(x) = -(x - 2)^2$
b) $f(x) = \frac{1}{2}x^2 + 2$
c) $f(x) = (x + 2)^2 - 2$

9. For each of the following, state the equation of a parabola congruent to $y = x^2$ with the given property.

a) The graph is 2 units to the right of the graph of $y = x^2$.
b) The graph is 4 units to the left of the graph of $y = x^2$.
c) The graph is 4 units to the left and 5 units down from the graph of $y = x^2$.
d) The graph is vertically compressed by a factor of 4.
e) The graph is vertically stretched by a factor of 2 and is 4 units to the left of the graph of $y = x^2$.
f) The graph is vertically stretched by a factor of 3 and is 2 units to the right and 1 unit down from the graph of $y = x^2$.

10. T For each of the following, state the condition on a and k such that the parabola $y = a(x - h)^2 + k$ has the given property.

a) The parabola intersects the x-axis at two distinct points.
b) The parabola intersects the x-axis at one point.
c) The parabola does not intersect the x-axis.

11. The acceleration due to gravity, g, is 9.8 m/s^2 on Earth, 3.7 m/s^2 on Mars, 10.5 m/s^2 on Saturn, and 11.2 m/s^2 on Neptune. The height, $h(t)$, of an object, in metres, dropped from above each surface is given by $h(t) = -0.5gt^2 + k$.
A

a) Describe how the graphs will differ for an object dropped from a height of 100 m on each of the four planets.

b) On which planet will the object be moving fastest when it hits the surface?

c) On which planet will it be moving slowest?

12. Describe how the x- and y-coordinates of the given quadratic functions differ from the x- and y-coordinates of corresponding points of $y = x^2$.
C

a) $y = (x + 7)^2$

b) $y = x^2 + 7$

c) $y = -2(x - 4)^2$

d) $y = -\frac{1}{2}x^2 - 4$

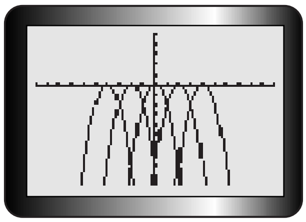
Extending

13. Predict what the graphs of each group of equations would look like. Check your predictions by using graphing technology.

a) $f(x) = 10x^2$
$f(x) = 100x^2$
$f(x) = 1000x^2$
$f(x) = 10000x^2$

b) $f(x) = 0.1x^2$
$f(x) = 0.01x^2$
$f(x) = 0.001x^2$
$f(x) = 0.0001x^2$

14. a) If $y = x^2$ is the base curve, write the equations of the parabolas that produce the following pattern shown on the calculator screen below. The scale on both axes is 1 unit per tick mark.



b) Create your own pattern using parabolas, and write the associated equations. Use $y = x^2$ as the base parabola.