1.2 Comparing Rates of Change in Linear and Quadratic Functions

GOAL

Identify and compare the characteristics of linear and quadratic functions.

YOU WILL NEED

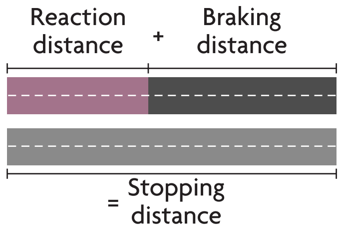
- graph paper

LEARN ABOUT the Math

Stopping distance is the distance a car travels from the time the driver decides to apply the brakes to the time the car stops. Stopping distance is split into two types of distances.

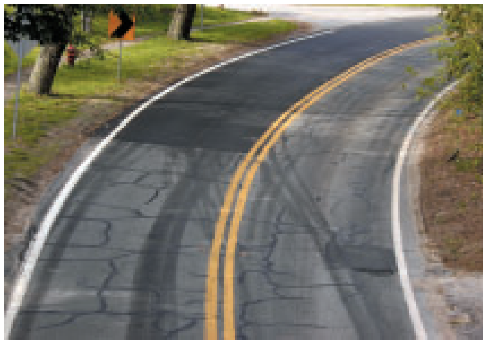
1. **Reaction Distance**: the distance the car travels from the time you first think "stop" to the time your foot hits the brake
2. **Braking Distance**: the distance the car travels from the time you first apply the brakes to the time the car stops

Stopping Distance = Reaction Distance + Braking Distance



The table shows some average values of these three distances, in metres, for different speeds.

Speed (km/h)	Speed (m/s)	Reaction Distance (m)	Braking Distance (m)	Stopping Distance (m)
0	0.00	0.00	0.00	0.00
20	5.56	8.33	1.77	10.11
40	11.11	16.67	7.10	23.76
60	16.67	25.00	15.96	40.96
80	22.22	33.33	28.38	61.71
100	27.78	41.67	44.35	86.01



? How does the relationship between speed and reaction distance compare with the relationship between speed and braking distance?

EXAMPLE 1 Representing and comparing linear and quadratic functions

Compare the relationship between speed and reaction distance with the relationship between speed and braking distance.

Kayla's Solution

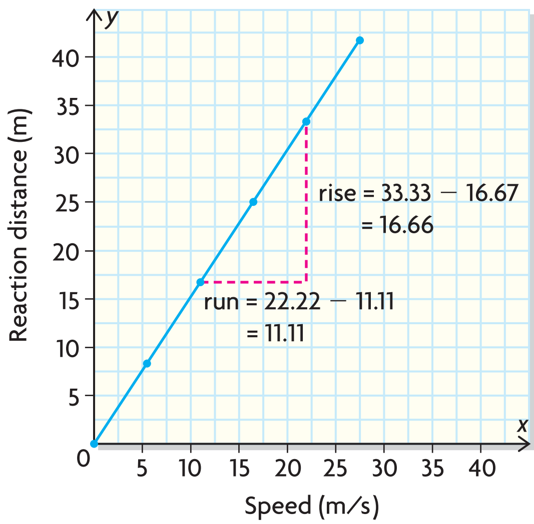
Part 1: Comparing speed and reaction distance

Speed (m/s)	Reaction Distance (m)	First Differences
0.00	0.00	
5.56	8.33	$8.33 - 0 = 8.33$
11.11	16.67	$16.67 - 8.33 = 8.34$
16.67	25.00	$25.00 - 16.67 = 8.33$
22.22	33.33	$33.33 - 25.00 = 8.33$
27.78	41.67	$41.67 - 33.33 = 8.34$

I calculated the first differences of the dependent variable, reaction distance.

This relationship is linear.

The first differences appear to be close to a constant, about 8.33.



I drew a scatter plot and a line of best fit to help me determine an equation for this relationship.

$$\text{slope} = \frac{16.66}{11.11} \doteq 1.5 \text{ and } y\text{-intercept} = 0.$$

I used the slope of the line and the y-intercept to determine that the equation of the line of best fit is $y = 1.5x$.

So, $y = 1.5x$ is the equation that relates speed to the reaction distance.

The degree of this relation is 1.

degree

the degree of a polynomial with a single variable, say, x, is the value of the highest exponent of the variable. For example, for the polynomial $5x^3 - 4x^2 + 7x - 8$, the highest power or exponent is 3; the degree of the polynomial is 3

$$y = f(x) = 1.5x$$

Since there is a unique reaction distance for each value of speed, the reaction distance is a function of speed. I can express this relationship in **function notation**.

function notation

$f(x)$ is called function notation and represents the value of the dependent variable for a given value of the independent variable, x

Part 2: Comparing speed and braking distance

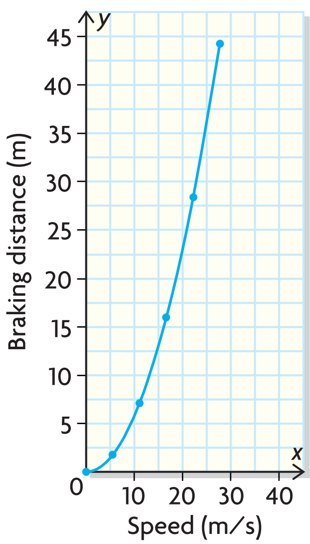
Speed (m/s)	Braking Distance (m)	First Differences	Second Differences
0.00	0.00		
		1.77	
5.56	1.77		3.56
		5.33	
11.11	7.10		3.53
		8.86	
16.67	15.96		3.56
		12.42	
22.22	28.38		3.55
		15.97	
27.78	44.35		

I calculated the first differences. They are not constant, so the relationship is nonlinear.

I calculated the second differences. They are almost the same value, so the relationship is quadratic.

Communication *Tip*

$f(x)$ is read as "f of x" or "f at x." The symbols $f(x)$, $g(x)$, and $h(x)$ are often used to name functions, but other letters may be used. When working on a problem, it is sometimes easier to choose letters that match the quantities in the problem. For example, use $v(t)$ to write velocity as a function of time.



I drew a scatter plot and a smooth curve to help me find the equation of this quadratic relationship.

$y = ax^2$

$a = \frac{y}{x^2}$

$a = \frac{7.10}{11.11^2}$

$a \doteq 0.0575$

$y = 0.0575x^2$

is the equation that relates speed to braking distance.

I used $y = ax^2$ since it looks like the parabola's vertex is (0, 0). I chose the point (11.11, 7.10) and substituted $y = 7.10$ and $x = 11.11$ into the equation to solve for a.

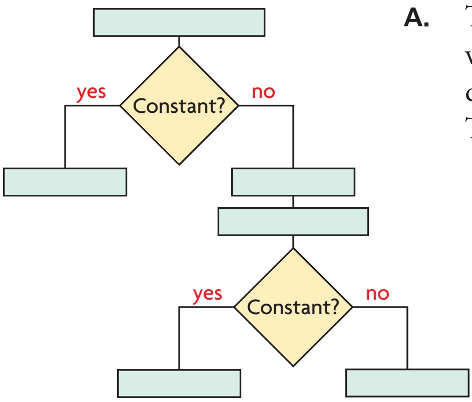
The degree of this relation is 2.

The highest power of the variable, x, is 2.

$y = g(x) = 0.055x^2$

Since there is a unique breaking distance for each value of speed, braking distance is a function of speed. I wrote the equation using function notation using the letter g as the name of the function since I already named the stopping distance function f.

Reflecting



A. The chart at the left summarizes the sequence of steps used to decide whether data for a function are linear, quadratic, or neither. The diamonds in the chart represent decisions that have yes-or-no answers. The rectangles represent steps in the sequence.

i) Copy the chart. Then complete it by matching each empty rectangle with one of the six phrases.

1. calculate second differences
2. linear
3. neither quadratic nor linear
4. nonlinear
5. calculate first differences
6. quadratic

ii) Use the steps in the chart to determine whether stopping distance is a linear or quadratic relation. The data for stopping distance are provided in the table on page 17.

B. What is the relationship between the differences in a table of values and the degree of a function?

C. How can you tell whether a function is linear or quadratic if you are given
- a table of values?
- a graph?
- an equation?

APPLY the Math

EXAMPLE 2 Representing volume as a function of time

Water is poured into a tank at a constant rate. The volume of water in the tank is measured every minute until the tank is full. The measurements are recorded in the table.

Time (min)	0	1	2	3	4	5	6	7
Volume (L)	0.0	1.3	2.6	3.9	5.2	6.5	7.8	9.1

a) Use difference tables to determine whether the volume of water poured into the tank, $V(t)$, is a linear or quadratic function of time. Explain.

b) State the domain and range using set notation.

Brian's Solution

a)

Time (min)	0	1	2	3	4	5	6	7
Volume (L)	0.0	1.3	2.6	3.9	5.2	6.5	7.8	9.1
First Differences		1.3	1.3	1.3	1.3	1.3	1.3	1.3

Every minute, an additional 1.3 L of water is added to the tank. Since the first differences are constant, $V(t)$ is a linear function.

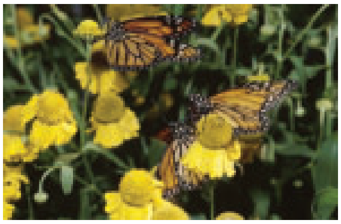
b) Domain $= \{t \in \mathbf{R} \mid 0 \leq t \leq 7\}$

It takes 7 min to fill the tank, and time can't be negative.

Range $= \{V(t) \in \mathbf{R} \mid 0 \leq V(t) \leq 9.1\}$

The tank holds a minimum of 0 L when empty and a maximum of 9.1 L of water when full.

EXAMPLE 3 Representing distance travelled as a function of time



A migrating butterfly travels about 128 km each day. The distance it travels, $g(t)$, in kilometres, is a function of time, t, in days.

a) Write an equation using function notation to represent the distance a butterfly travels in t days.

b) State the degree of this function and whether it is linear or quadratic.

c) Use the function to calculate the distance the butterfly travels in 20 days.

d) What are the domain and range of this function, assuming that a butterfly lives 40 days, on average? Express your answers in set notation.

Rashmika's Solution

a) $g(t) = 128t$

In this equation, t is the number of days and $g(t)$ is the total distance travelled for the given value of t.

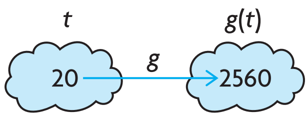
b) The degree is 1.

The function is linear.

The highest exponent of t is 1. This means the function $g(t)$ is linear.

c)
$$\begin{aligned} g(t) &= 128t \\ g(20) &= 128(20) \\ &= 2560 \end{aligned}$$

Substitute $t = 20$ into the function $g(t)$.



The butterfly travels 2560 km in 20 days.

d) Domain $= \{t \in \mathbf{R} \mid 0 \le t \le 40\}$

The maximum number of days is 40.

Range $= \{g(t) \in \mathbf{R} \mid 0 \le g(t) \le 5120\}$

The maximum distance is $g(40) = 5120$ km.

In this case, neither distance nor time can be negative.

If you know the equation of a function, its degree indicates whether it is linear or quadratic.

EXAMPLE 4 Connecting degree to type of function

State the degree of each function, and identify which functions are linear and which are quadratic.

a) $k(x) = 3x(x + 1)$

b) $m(x) = (x + 2)^2 - x^2$

Akiko's Solution

a) $k(x) = 3x(x + 1)$
$= 3x^2 + 3x.$

I used the distributive property to expand.

The degree is 2.

The highest exponent of x is 2.

The function is quadratic.

b) $m(x) = (x + 2)^2 - x^2$
$= (x + 2)(x + 2) - x^2$
$= x(x + 2) + 2(x + 2) - x^2$
$= x^2 + 2x + 2x + 4 - x^2$
$= x^2 + 4x + 4 - x^2$
$= 4x + 4$

I used the distributive property to expand.

I simplified by collecting like terms.

The degree is 1.

The highest exponent of x is 1.

The function is linear.

Study Aid

- For help, see Essential Skills Appendix, A-9.

In Summary

Key Ideas

- Linear functions have constant first differences, a degree of 1, and graphs that are lines.
- Quadratic functions have constant second differences, a degree of 2, and graphs that are parabolas.

Need to Know

- $f(x)$ is called function notation and is used to represent the value of the dependent variable for a given value of the independent variable, x.
- The degree of a function is the highest power of the independent variable.

CHECK Your Understanding

1. A ball is dropped from a distance 10 m above the ground. The height of the ball from the ground is measured every tenth of a second, resulting in the following data:

Time (s)	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8
Height (m)	10.00	9.84	9.36	8.56	7.44	6.00	4.24	2.16	0.00

   a) Use difference tables to determine whether distance, $d(t)$, is a linear or quadratic function of time. Explain.

   b) What are the domain and range of this function? Express your answer in set notation.

2. A math textbook costs \$60.00. The number of students who need the book is represented by x. The total cost of purchasing books for a group of students can be represented by the function $f(x)$.

   a) Write an equation in function notation to represent the cost of purchasing textbooks for x students.

   b) State the degree of this function and whether it is linear or quadratic.

   c) Use your equation to calculate the cost of purchasing books for a class of 30 students.

   d) What are the domain and range of this function, assuming that books can be purchased for two classes of students? Assume that the maximum number of students in a class is 30. Express your answers in set notation.

3. State the degree of each function, and identify which are linear and which are quadratic.

   a) $f(x) = -7 + 2x$

   b) $g(x) = 3x^2 + 5$

   c) $g(x) = (x - 4)(x - 3)$

   d) $3x - 4y = 12$

PRACTISING

4. Use difference tables to determine whether the data represent a linear or quadratic relationship.

Time (s)	0	1	2	3	4	5
Height (m)	0	15	20	20	15	0

Time (h)	**Bacteria Count**
0	12
1	23
2	50
3	100

5. **K** The population of a bacteria colony is measured every hour and results in the data shown in the table at the left. Use difference tables to determine whether the number of bacteria, $n(t)$, is a linear or quadratic function of time. Explain.

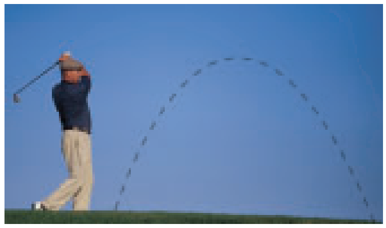
6. State the degree of each function and whether it is linear or quadratic.
   a) $f(x) = -4x(x - 1) - x$
   b) $m(x) = -x^2 + (x + 3)^2$
   c) $g(x) = 3x^2 + 35$
   d) $g(x) = 3(x - 5)$

7. **T** A function has the following domain and range:

$$\text{Domain} = \{t \in \mathbf{R} \mid -3 \leq t \leq 5\}$$
$$\text{Range} = \{g(t) \in \mathbf{R} \mid 0 \leq g(t) \leq 10\}$$

   a) Draw a sketch of this function if it is linear.
   b) Draw a sketch of this function if it is quadratic.

8. **A** A golf ball is hit and its height is given by the equation $h = 29.4t - 4.9t^2$, where t is the time elapsed, in seconds, and h is the height, in metres.
   a) Write an equation in function notation to represent the height of the ball as a function of time.
   b) State the degree of this function and whether the function is linear or quadratic.
   c) Use difference tables to confirm your answer in part (b).
   d) Graph the function where $\{t \in \mathbf{R} \mid t \geq 0\}$.
   e) At what time(s) is the ball at its greatest height? Express the height of the ball at this time in function notation.
   f) At what time(s) is the ball on the ground? Express the height of the ball at this time in function notation.



9. **C** List the methods for determining whether a function is linear or quadratic. Use examples to explain the advantages and disadvantages of each method.

Extending

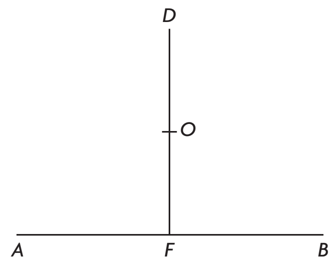
10. A baseball club pays a vendor \$50 per game for selling bags of peanuts for \$2.50 each. The club also pays the vendor a commission of \$0.05 per bag.
    a) Determine a function that describes the income the vendor makes for each baseball game. Define the variables in your function.
    b) Determine a function that describes the revenue the vendor generates each game for the baseball club. Define the variables in your function.
    c) Determine the number of bags of peanuts the vendor must sell before the baseball club makes a profit from his efforts.

Curious Math

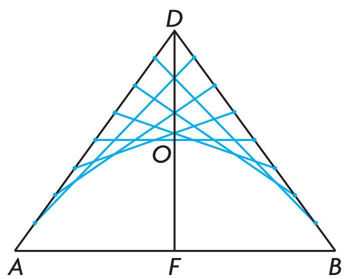
Using Straight Lines to Draw a Parabola

How can you use a series of lines to create a parabola?

1. Draw lines AB and FD so that line AB is perpendicular to FD and $AF = FB$. Mark O on line DF so that $DO = OF$.

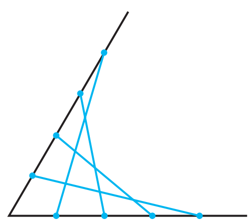


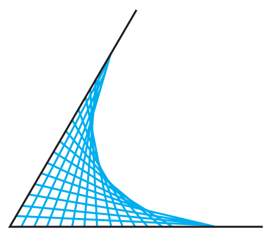
2. Draw lines DA and DB.
3. Divide DA into 8 equal parts. Divide DB into 8 equal parts.
4. Connect the new points as shown in the diagram to create additional lines. What do you notice?



5. Divide lines DA and DB into smaller equal parts. Connect the new points as in step 4. What do you notice?

Here are two similar sketches. The one on the left uses few lines, and the one on the right uses many. As the number of lines used increases, the shape of the parabola becomes more noticeable.





Why do you think this series of lines creates a parabola?